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Set-Theoretical Solutions of the n-Simplex Equation

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Abstract—The n-simplex equation was introduced by Zamolodchikov as a generalization of the Yang–Baxter equation which becomes the 2-simplex equation in this terms. In the present article, we suggest general approaches to construction of solutions of the n-simplex equation, describe certain types of solutions, and introduce an operation that allows us to construct, under certain conditions, a solution of the $(n+m+k)$-simplex equation from solutions of the $(n+k)$-simplex equation and $(m+k)$-simplex equation. We consider the tropicalization of rational solutions and discuss its generalizations. We prove that a solution of the n-simplex equation on G can be constructed from solutions of this equation on H and K if G is an extension of a group H by a group K. We also find solutions of the parametric Yang–Baxter equation on H with parameters in K. We introduce ternary algebras for studying the 3-simplex equation and present examples of such algebras that provide us with solutions of the 3-simplex equation. We find all elementary verbal solutions of the 3-simplex equation on a free group.

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1. INTRODUCTION

A solution of the quantum Yang–Baxter equation is a linear mapping $R : V \otimes V \to V \otimes V$ satisfying the condition

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.1}$$

where V is a vector space and each mapping $R_{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$ acts as R on the ith and jth factor of the tensor product and as the identity mapping on the remaining factor. A mapping R that satisfies the Yang–Baxter equation is called an Yang–Baxter mapping, see [9]. The Yang–Baxter equation is also called the triangle equation or the 2-simplex equation. This is one of the basic equations of mathematical physics and low-dimensional topology. It forms a foundation for theory of quantum groups, statistical physics, theory of braid groups, knot theory, and other directions in mathematics and physics. The Yang–Baxter equation was introduced in Yang's article [42] on the many-body problem. Baxter [6] used this equation for studying solvable vertex models in statistical mechanics, where it served as a condition for transfer matrices to commute. The Yang–Baxter equation arises in decomposition of S-matrices and $(1+1)$-dimensional quantum field theory, see Zamolodchikov's articles [43, 44]. We

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also mention an important role of the Yang–Baxter equation in the quantum inverse scattering method for integrable systems [39, 41].

It is not difficult to see that the Yang–Baxter equation is equivalent to a system of n^6 cubic algebraic equations in n^4 variables, where $n = \dim V$. In particular, if $n = 2$ then we obtain a system of 64 equations in 16 variables with no obvious solution. A complete description of solutions of the Yang–Baxter equation for $n = 2$ can be found in [29] (see also [20]).

Drinfel'd [15] suggested to study set-theoretical solutions, i.e., solutions such that the vector space V is spanned by a set X and R is the linear mapping induced by a mapping $R : X \times X \to X \times X$. Such pairs (X, R) are called set-theoretical solutions (or, simply, solutions) of the Yang–Baxter equation. Set-theoretical solutions are connected with I-type, Biberbach, Garside groups, etc. Involutive set-theoretical solutions of the Yang–Baxter equation were studied in [17].

It is easy to see that, for every set X, the equality $P(x, y) = (y, x)$ for $x, y \in X$ defines a solution P of the Yang–Baxter equation. On the other hand, if R is a solution of the Yang–Baxter equation then the mapping $S = PR$ satisfies the braid relation

$$(S \times \mathrm{id})(\mathrm{id} \times S)(S \times \mathrm{id}) = (\mathrm{id} \times S)(S \times \mathrm{id})(\mathrm{id} \times S),$$

which is a relation in the braid group B_3. From the point of view of topology, the braid relation is the third Reidemeister move of planar diagrams of links. In the 1980s, Joyce [22] and Matveev [34] introduced quandles as invariants of knots and links. They proved that each quandle provides us with a set-theoretical solution of the braid relation. A number of articles (see, for example, [9, 16, 32] and the references in those articles) is devoted to set-theoretical solutions of the Yang–Baxter equation and the braid relation.

In [35], solutions of the parametric Yang–Baxter equation were constructed. In [10], a method was suggested for constructing linear parametric solutions of the Yang–Baxter equation from nonlinear Darboux transforms of Lax operators.

In 2006, Rump [36, 37] introduced braces for studying involutive set-theoretical solutions of the Yang–Baxter equation. Notice that the axioms of braces can be traced back to the lectures of Kurosh [31] in the 1960s. In 2014, Cedó, Jespers, and Okniński [13] suggested another approach to defining braces. In 2017, Guarnieri and Vendramin [19] introduced skew braces that give rise to noninvolutive solutions of the Yang–Baxter equation.

A solution $R(x, y) = (\sigma_y(x), \tau_x(y))$ of the Yang–Baxter equation is said to be nondegenerate if σ_x and τ_x are invertible for every $x \in X$. A solution is said to be square-free if $R(x, x) = (x, x)$ for every $x \in X$. A solution $R(x, y) = (\sigma_y(x), \tau_x(y))$ determines a 2-groupoid $(X; \cdot, *)$, i.e., a set endowed with two binary operations $\cdot, * : X \times X \to X$ such that $x \cdot y = \sigma_y(x)$ and $y * x = \tau_x(y)$. A solution (X, R) is said to be elementary if either $\sigma_y = \mathrm{id}$ for every $y \in X$ or $\tau_x = \mathrm{id}$ for every $x \in X$. Each elementary solution determines a self-distributive groupoid with the universe X. If an elementary solution is nondegenerate then the corresponding groupoid is a rack. On the other hand, every self-distributive groupoid provides us with a solution of the Yang–Baxter equation and every rack leads to a nondegenerate solution of the Yang–Baxter equation. Every quandle provides us with a nondegenerate square-free solution of the Yang–Baxter equation.

The 3-simplex equation (or the tetrahedron equation) was introduced by Zamolodchikov [43, 44] as a 3-dimensional generalization of the Yang–Baxter equation. The tetrahedron equation

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$

is a nonlinear relation in $V^{\otimes 6}$ for an endomorphism $R : V^{\otimes 3} \to V^{\otimes 3}$, where R_{ijk} acts as R on the ith, jth, and kth factors V of the tensor product and as the identity mapping on the remaining factors. The tetrahedron equation is popular in the theory of electric networks. In particular, the well-known star-triangle transform provides us with a rational solution of the tetrahedron equation. It is also used in the theory of 2-dimensional knots in a 4-dimensional sphere [28], serves as an analog of the third Reidemeister move in classical knot theory, and arises in the problem of coloring 2-faces of a 4-cube by elements of a set X [27].

Numerous solutions of the tetrahedron equation are already known. For example, Hietarinta [21] considered the set $\mathbb{Z}_d$ of integers modulo d as X and studied linear and affine mappings $X^n \to X^n$

that give rise to solutions of the n-simplex equation for $n = 2, 3, 4$. Linear solutions of the tetrahedron equation were described in [25].

The n-simplex equation is a generalization of both the Yang–Baxter equation and the tetrahedron equation. It was introduced in [7] and studied in [11, 12, 14, 21, 23, 38]. In the present article, we suggest general approaches to construction of solutions of the n-simplex equation, describe several types of solutions, and introduce an operation that, under certain conditions, allows us to construct a solution of the $(n+m+k)$-simplex equation from solutions of the $(n+k)$-simplex equation and $(m+k)$-simplex equation. We consider the tropicalization of rational solutions and discuss its generalizations. We prove that a solution of the n-simplex equation on G can be constructed from solutions of this equation on H and K if G is an extension of a group H by a group K. We also find solutions of the parametric Yang–Baxter equation on H with parameters in K. We introduce ternary algebras for studying the 3-simplex equation and present examples of such algebras that provide us with solutions of the 3-simplex equation. We find all elementary verbal solutions of the 3-simplex equation on a free group.

We briefly describe the contents of the article. In Section 2, we recall the connection between set-theoretical solutions of the Yang–Baxter equation and certain algebras (self-distributive groupoids, racks, and quandles). We formulate conditions that guarantee that a set-theoretical solution of the Yang–Baxter equation arises from an algebra with two binary operations. We also recall the definitions and some properties of braid groups and virtual braid groups.

In Section 3, we consider the general form of the n-simplex equation, describe an algorithm for writing the n-simplex equation for a prescribed $n \geq 2$, and formulate results on general solutions. We present a definition of the classical n-simplex equation that is similar to the classical Yang–Baxter equation.

As is known, if (X, R) is a solution of the Yang–Baxter equation and $P_{12}(x, y) = (y, x)$ then $P_{12}RP_{12}$ is a solution of the Yang–Baxter equation too. In Proposition 4.4, we prove a similar assertion for the n-simplex equation, i.e., we construct a permutation P such that PRP satisfies this equation.

For a natural k, we introduce the k-amalgam of solutions of the $(n+k)$-simplex and $(m+k)$-simplex equations and find sufficient conditions (see Theorem 4.7) for the k-amalgam to be a solution of the $(n+m+k)$-simplex equation. We further apply this theorem to linear solutions.

In Proposition 4.16, we prove that the 0-amalgam of linear solutions of the n-simplex and m-simplex equations is a solution of the $(n+m)$-simplex equation. If the 1-amalgam of these solutions is defined then it is a solution of the $(n+m-1)$-simplex equation. We also prove that a specialization of a solution of the n-simplex equation at a fixed point (if such a point exists) is a solution of the $(n-1)$-simplex equation.

In Section 5, we consider a class of rational solutions of the n-simplex equation and define their tropicalization. We show that tropicalization of solutions of this class provides us with piecewise-linear solutions of the n-simplex equation. In particular, we obtain a linear solution of the tetrahedron equation from the electric solution. We further generalize the tropicalization procedure to other algebraic systems. As an example, we construct a homomorphism $h : \mathbb{R} \to \mathcal{D}'[\mathbb{R}^m]$ from the field $\mathbb{R}$ or reals to the linear space $\mathcal{D}'[\mathbb{R}^m]$ of compactly supported distributions on $\mathbb{R}^m$ with the addition and convolution operations. This homomorphism allows us to construct solutions $(\mathcal{D}'[\mathbb{R}^m], R^h)$ of the n-simplex equation from rational solutions of this equation over $\mathbb{R}$.

In Section 6, we consider group extensions. Let G be a group and let H be a normal subgroup. Assume that a structure of a self-distributive groupoid can be defined on G and certain conditions on the quotient group G/H hold. We show that it is possible to construct a solution of the Yang–Baxter equation on H with parameters in G/H. In particular, this is possible if K is a trivial self-distributive groupoid. We generalize these results to the n-simplex equation for an arbitrary n. In Subsection 6, we use known representations of braid groups and construct two types of solutions of the Yang–Baxter equation with parameters in an Abelian group, see Theorem 6.6. We also show that, for every extension G of a group H by a group K and all solutions (H, R) and (K, T) of the n-simplex equation, it is possible to define a natural solution of the n-simplex equation on G.

In Section 8, we consider the category of solutions of the n-simplex equation and prove that the limit of solutions in this category is a solution too. In particular, we show (see Corollary 8.2) that solutions $(\mathbb{Z}_{p^k}, R_k)$, $k = 1, 2, \dots$, of the n-simplex equation give rise to a solution of the n-simplex equation on p-adic numbers if certain conditions hold.

In Section 9, by analogy with racks, bi-racks, and skew braces introduced for studying the Yang–Baxter equation, we introduce ternars and 3-ternoids. These are algebras with a single and three ternary operations respectively. We show that they are connected with solutions of the tetrahedron equation, see Proposition 9.1 and corollaries to it. We also introduce algebras with four binary operations and relate them to elementary solutions of the tetrahedron equation, see Propositions 9.6 and 9.10.

In Section 10, we introduce verbal solutions. We describe all elementary verbal solutions of the tetrahedron equation on free non-Abelian groups.

We also formulate open questions for further study.

We use the following notation. If G is a group and $a, b \in G$ then $a^b = b^{-1}ab$ (i.e., the elements a and a^b are conjugate with respect to b) and $[a, b] = a^{-1}b^{-1}ab$ (i.e., $[a, b]$ is the commutator of a and b). If $f, g : X \to X$ then the composition fg is defined by the rule $(fg)(x) = f(g(x))$.

2. PRELININARIES

In the present section, we recall necessary basic facts on the Yang–Baxter equation. We describe the relations between its solutions and classical and virtual braid groups, as well as certain groupoids (racks and quandles). We also present examples of such solutions.

2.1. *The Yang–Baxter equation and racks.*

Lemma 2.1. *We consider a nonempty set X and mappings $f, g : X^2 \to X$. We put $R(x, y) = (f(x, y), g(x, y))$. The obtained mapping $R : X^2 \to X^2$ is a solution of the Yang–Baxter equation if and only if, for all $x, y, z \in X$,*

$$\begin{aligned} f(f(x,y),z) &= f(f(x,g(y,z)),f(y,z)),\\ f(g(x,y),g(f(x,y),z)) &= g(f(x,g(y,z)),f(y,z)),\\ g(g(x,y),g(f(x,y),z)) &= g(x,g(y,z)). \end{aligned}$$

Example 2.2.

1. Let A be an Abelian group and let $a, b \in A$. We put
$$R(x, y) = (x + a, y + b)$$
for all $x, y \in A$. Then R is a solution of the Yang–Baxter equation on A.
2. For every group G, we put
$$R(x, y) = (xy^2, xy^{-1}x^{-1})$$
for all $x, y \in G$. Then R is a solution of the Yang–Baxter equation on G.
3. Let G be a group and let $\varphi \in \mathrm{Aut}(G)$ be an automorphism. We put
$$R(x, y) = (\varphi(x), xy\varphi(x)^{-1})$$
for all $x, y \in G$. Then R is a solution of the Yang–Baxter equation on G.

A *groupoid* is a nonempty set endowed with a single binary operation. For a positive integer k with $k > 1$, a *k-groupoid* is a nonempty set endowed with k binary operations. Let (X, R) be a set-theoretical solution of the Yang–Baxter equation and let $R(x, y) = (\sigma_y(x), \tau_x(y))$, where $x, y \in X$. We say that R is a *nondegenerate* solution if both σ_x and τ_x are invertible for every $x \in X$. We say that R is a *square-free* solution if $R(x, x) = (x, x)$ for every $x \in X$.

A *quandle* is a groupoid Q with a binary operation $(x, y) \mapsto x * y$ that satisfies the following axioms:

(Q1) we have $x * x = x$ for every $x \in Q$,

(Q2) for all $x, y \in Q$, there exists a unique $z \in Q$ with $x = z * y$,

(Q3) we have $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

A groupoid is said to be *self-distributive* if it satisfies axiom (Q3). A *rack* is a groupoid that satisfies axioms (Q2) and (Q3).

Example 2.3. Let G be a group.

1. We define $a * b = b^{-1}ab$. Then G endowed with the operation $*$ is a quandle. We call it the *conjugation quandle* of G and denote by $\mathrm{Conj}(G)$.
2. We define $a * b = ba^{-1}b$. Then G endowed with the operation $*$ is a quandle. We call it the *core quandle* of G and denote by $\mathrm{Core}(G)$. If $G = \mathbb{Z}/n\mathbb{Z}$ then this quandle is said to be *dihedral* and is denoted by R_n.
3. Let $\phi \in \mathrm{Aut}(G)$. We define $a * b = \phi(ab^{-1})b$. Then G endowed with the operation $*$ is a quandle. We call it the *generalized Alexander quandle* of G with respect to ϕ and denote by $\mathrm{Alex}(G, \phi)$.

A quandle Q is said to be *trivial* if $x * y = x$ for all $x, y \in Q$. Unlike groups, the universe of a trivial quandle need not be a singleton. We denote by T_n a trivial quandle whose universe is an n-element set. By T we mean an arbitrary trivial quandle.

We define a mapping $S_x : Q \to Q$. We put

$$S_x(y) = y * x.$$

Notice that axioms (Q2) and (Q3) holds if and only if S_x is an automorphism of Q for every $x \in Q$. Such automorphisms are said to be *inner*. They form a group which is denoted by $\mathrm{Inn}(X)$. A quandle Q is said to be *connected* if the action of its group of inner automorphisms is transitive on Q. For example, the dihedral quandle R_n is connected if n is odd and is disconnected if n is even. A quandle X is said to be *involutive* if $S_x^2 = \mathrm{id}_Q$ for every $x \in Q$. For example, each core quandle is involutive for every group.

Let (X, R) with $R(x, y) = (\sigma_y(x), \tau_x(y))$ be a solution. We define a 2-groupoid $(X, \cdot, *)$. We put $x \cdot y = \sigma_y(x)$ and $y * x = \tau_x(y)$. If either $\sigma_y = \mathrm{id}$ for every $y \in X$ or $\tau_x = \mathrm{id}$ for every $x \in X$ then (X, R) is said to be *elementary*. Each elementary solution determines a self-distributive groupoid on X. If this solution is nondegenerate then the groupoid is a rack. On the other hand, each self-distributive groupoid determines an elementary solution of the Yang—Baxter equation and each rack determines a nondegenerate solution of the Yang—Baxter equation; moreover, a quandle determines a nondegenerate square-free solution of the Yang—Baxter equation.

From Lemma 2.1 we deduce the following assertion that establishes a connection between solutions and 2-groupoids.

Lemma 2.4. *Let $(X, \cdot, *)$ be a 2-groupoid and let $R : X \times X \to X \times X$, where*

$$R(x, y) = (x \cdot y, y * x) \text{ for all } x, y \in X.$$

1. *The pair (X, R) is a set-theoretical solution of the Yang—Baxter equation if and only if the conditions*

$$\begin{aligned} (x \cdot y) \cdot z &= (x \cdot (z * y)) \cdot (y \cdot z), \\ (y * x) \cdot (z * (x \cdot y)) &= (y \cdot z) * (x \cdot (z * y)), \\ (z * (x \cdot y)) * (y * x) &= (z * y) * x \end{aligned}$$

   hold for all $x, y, z \in X$.
2. *If $x \cdot y = y$ for all $x, y \in X$ then (X, R) is a set-theoretical solution of the Yang—Baxter equation if and only if the operation $*$ is self-distributive, i.e., we have*

$$(z * x) * (y * x) = (z * y) * x$$

   for all $x, y, z \in X$.
3. *If $y * x = y$ for all $x, y \in X$ then (X, R) is a set-theoretical solution of the Yang—Baxter equation if and only if the operation $\cdot$ is self-distributive, i.e., we have*

$$(z \cdot y) \cdot x = (z \cdot x) \cdot (y \cdot x)$$

   for all $x, y, z \in X$.

Example 2.5. On the set $\mathbb{R}$ of reals, we define a binary operation. We put

$$a * b = \max\{a, b\}$$

for all $a, b \in \mathbb{R}$. Then $(\mathbb{R}, *)$ is a self-distributive groupoid. This is a commutative groupoid that is not a rack. Each element $a \in \mathbb{R}$ is idempotent, i.e., we have $a * a = a$. We define $R(x, y) = (x, y * x)$. Then R is a degenerate solution of the Yang–Baxter equation on $\mathbb{R}$. If we consider min instead of max then we obtain a similar result.

As is proven in [40, Theorem 2.3] (see also [19, Proposition 3.7] and [1, Proposition 5.2]), every nondegenerate solution of the Yang–Baxter equation is conjugate to an elementary solution.

Proposition 2.6. *Let $R(x, y) = (\sigma_y(x), \tau_x(y))$ for all $x, y \in X$ and let R be a left nondegenerate solution of the Yang–Baxter equation, i.e., for every $y \in X$, let σ_y be a bijective mapping. Then R is conjugate to the elementary solution*

$$R'(x, y) = (\sigma_x(\tau_{\sigma_y^{-1}(x)}(y)), y).$$

If, for all $a, b \in X$, there exists $x \in X$ such that

$$\tau_{\sigma_x^{-1}(a)}(x) = \sigma_a^{-1}(b) \tag{2.1}$$

then this solution is nondegenerate.

2.2. *Braid and virtual braid groups.* For $n \geq 2$, the *braid group* B_n on n strands is presented by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, 2, \ldots, n-2, \tag{2.2}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2. \tag{2.3}$$

The *virtual braid group* VB_n on n strands was introduced in [26]. This group is presented by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and $\rho_1, \rho_2, \ldots, \rho_{n-1}$ and relations (2.2)–(2.3) of the corresponding braid group and the additional relations

$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \qquad i = 1, 2, \ldots, n-2, \tag{2.4}$$

$$\rho_i\rho_j = \rho_j\rho_i, \qquad |i-j| \geq 2, \tag{2.5}$$

$$\rho_i^2 = 1, \qquad i = 1, 2, \ldots, n-1, \tag{2.6}$$

$$\sigma_i\rho_j = \rho_j\sigma_i, \qquad |i-j| \geq 1, \tag{2.7}$$

$$\rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1}, \qquad i = 1, 2, \ldots, n-2. \tag{2.8}$$

The elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ generate a subgroup that is isomorphic to B_n. Relations (2.4)–(2.6) are relations of the symmetric group S_n. The elements $\rho_1, \rho_2, \ldots, \rho_{n-1}$ generate S_n. Thus, the group VB_n is generated by the braid group B_n and the symmetric group S_n. Relations (2.7)–(2.8) are called the *mixed* relations of the virtual braid group.

3. QUANTUM AND CLASSICAL n-SIMPLEX EQUATIONS

3.1. *Quantum n-simplex equation.* We present the quantum n-simplex equation for an arbitrary n. We recall the geometric interpretation of the Yang–Baxter equation. We consider straight lines l_1, l_2, and l_3 in the plain $\mathbb{R}^2$. Let l_1 intersect l_2 at the point R_{12} and l_3 at the point R_{13} and let R_{23} be the intersection point of l_2 and l_3. Assume that R_{12}, R_{13}, and R_{23} are pairwise distinct and form a nondegenerate triangle (2-simplex), see Fig. 1.

We consider the lexicographic order, i.e., let

$$R_{12} < R_{13} < R_{23}.$$

Then the Yang–Baxter equation is the equality between words, where the first word correspond to going around the simplex in the increasing order and the second word corresponds to the reverse order, i.e., we obtain

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

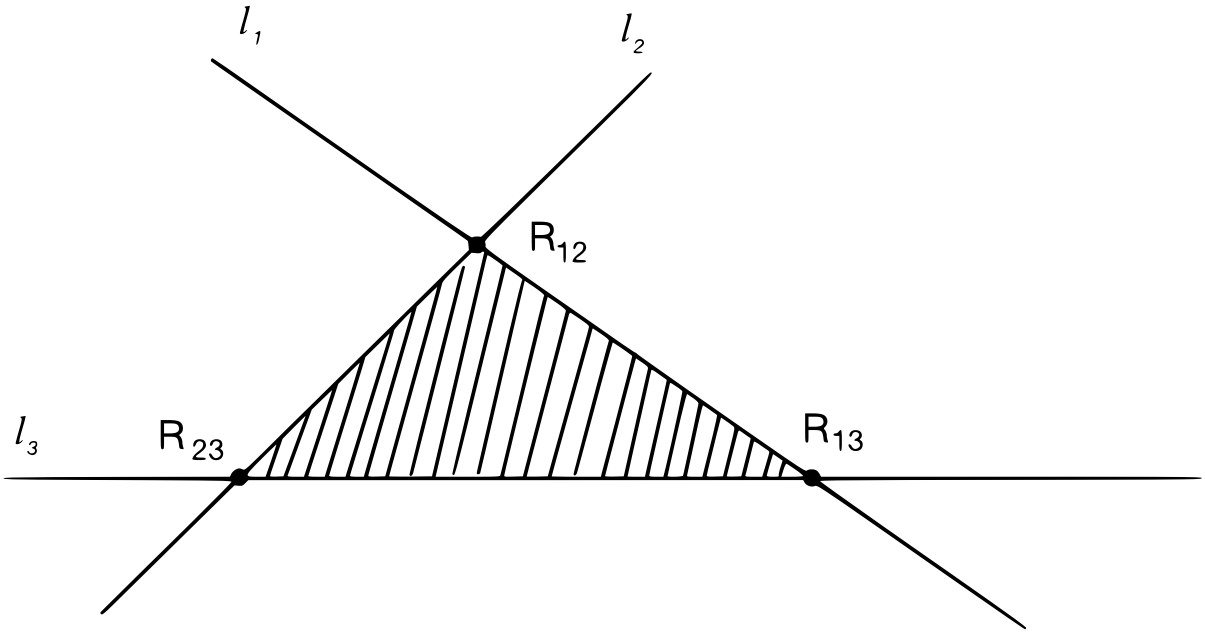


Fig. 1. Geometric interpretation of the Yang–Baxter equation

We pass to the tetrahedron (3-simplex) equation. We increment the subscripts by 3 for the straight lines in the plain from the Yang–Baxter equation and embed the plain into the 3-dimensional space. We choose a vertex that does not belong to the plain and construct straight lines l_1, l_2, and l_3 passing through this vertex and the vertices of the 2-simplex. We associate each vertex with the subscript that corresponds to straight lines passing through this vertex, see Fig. 2.

The tetrahedron equation is the equality between words, where the first word consists of vertices ordered according to the lexicographic order and the second word corresponds to the reverse order, i.e, we obtain

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}.$$

Using the same approach, we may write the 4-simplex equation as follows:

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}.$$

In general, the words on the left-hand and right-hand sides of the n-simplex equation are of length $n+1$ (this is the number of vertices of the n-simplex).

Assume that the n-simplex equation is constructed for $n \geq 3$. We rewrite it in the form

$$R_{\overline{1}}R_{\overline{2}}\cdots R_{\overline{n+1}} = R_{\overline{n+1}}\cdots R_{\overline{2}}R_{\overline{1}},$$

where $\overline{k} = (k_1, k_2, \ldots, k_{n+1}) \in \mathbb{N}^{n+1}$ is a multi-index. We construct the $(n+1)$-simplex equation. We define an operation $s_n : \mathbb{N} \to \mathbb{N}$. We put $s_n(k) = k + n + 1$. We extend this definition to multi-indices as follows:

$$s_n(\overline{k}) = (s_n(k_1), s_n(k_2), \ldots, s_n(k_{n+1})) \in \mathbb{N}^{n+1}.$$

In this notation, the $(n+1)$-simplex equation can be rewritten as follows:

$$R_{1,2,\ldots,n+1}R_{1,s_n(\overline{1})}R_{2,s_n(\overline{2})}\cdots R_{n+1,s_n(\overline{n+1})} = R_{n+1,s_n(\overline{n+1})}\cdots R_{2,s_n(\overline{2})}R_{1,s_n(\overline{1})}R_{1,2,\ldots,n+1}.$$

We may regards multi-indices in the n-simplex equation as columns in the *multi-index matrix* MI_n

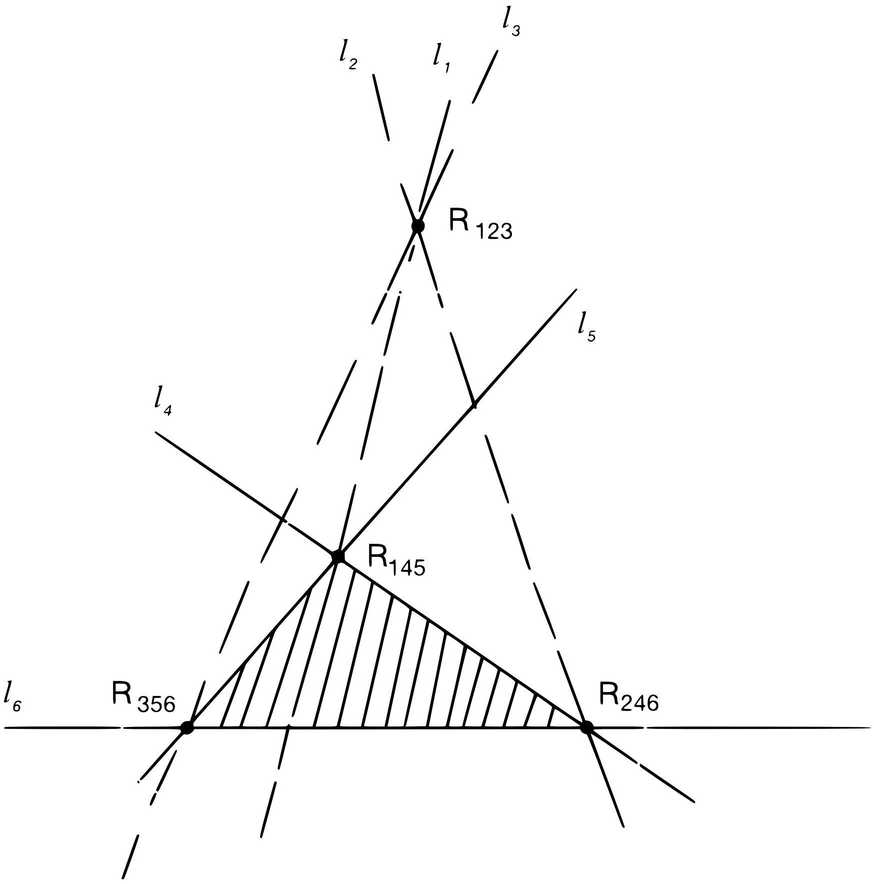


Fig. 2. Geometric interpretation of the tetrahedron equation

of size $(n+1)\times n$ that satisfies the recurrent relation

$$MI_n = \left(\begin{array}{c|c} 1 \quad 2 \quad\quad 3 & \cdots \; n \\ \hline 1 & \\ 2 & \\ \vdots & MI_{n-1} + (n) \\ n-1 & \\ n & \end{array}\right)$$

where (n) is the matrix of size $n\times(n-1)$ whose entries are equal to n. We present the explicit form of MI_n:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & n+1 & n+2 & \cdots & 2n-1 \\ 2 & n+1 & 2n & \cdots & 3n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 2n-2 & 3n-4 & \cdots & \frac{n(n+1)}{2} \\ n & 2n-1 & 3n-3 & \cdots & \frac{n(n+1)}{2} \end{pmatrix}.$$

By a *linear solution* of the n-simplex equation we mean a linear mapping $R : V^{\otimes n} \to V^{\otimes n}$ such that the equality

$$R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}} = R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}}$$

holds for two linear mappings $V^{\otimes N} \to V^{\otimes N}$, where $N = n(n+1)/2$. Here $R_{\overline{k}} : V^{\otimes N} \to V^{\otimes N}$ act as R on the copies of V with numbers in $\overline{k}$ and as the identity mapping on the remaining copies. Notice that $n+1$ is the number of vertices of the n-simplex and N is the number of its edges.

A *set-theoretical solution* of the n-simplex equation on a set X is a mapping $R : X^n \to X^n$ that satisfies the n-simplex equation.

For convenience, we also consider the 1-simplex equation

$$R_1 R_2 = R_2 R_1$$

(the equality for a pair of mappings on $X \times X$). In this case, by R we mean a mapping $R : X \to X$.

Notice that each mapping $R : X \to X$ is a solution of the 1-simplex equation because

$$R_1(x, y) = (R(x), y), \;\; R_2(x, y) = (x, R(y)), \;\; R_1 R_2(x, y) = R_2 R_1(x, y) = (R(x), R(y)).$$

3.2. *Classical n-simplex equation.* The equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

is called the quantum Yang—Baxter equation. Along with it, the classical Yang—Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

is considered (see [8]), where $[a, b] = ab - ba$.

If (V, R) is a linear solution and R can be represented in the form

$$R = 1 + \hbar r + O(\hbar^2)$$

then $r\colon V \to V$ is a solution of the classical Yang—Baxter equation.

The classical n-simplex equation is the equation

$$\sum_{\overline{1} \le \overline{i} < \overline{j} \le \overline{n+1}} [r_{\overline{i}}, r_{\overline{j}}] = 0, \tag{3.1}$$

where the sum it taken with respect to the lexicographic order on multi-indices. This definition agrees with the following assertion. Its proof consists in straightforward calculation.

Proposition 3.1. *If V is a vector space, a linear mapping $R\colon V^n \to V^n$ is a solution of the n-simplex equation, and there exists a mapping $r\colon V^n \to V^n$ such that*

$$R = 1 + \hbar r + O(\hbar^2)$$

then r is a solution of the classical n-simplex equation.

4. SOLUTIONS OF THE n-SIMPLEX EQUATION

In the sequel, we consider set-theoretical solutions only. We consider functions $R : X^n \to X^n$ that are defined on a subset $D \subset X^n$. We say that a pair (X, R) is a solution of the n-simplex equation if the n-simplex equation is valid at each point $\bar{x} \in X^N$, where the left-hand and right-hand sides are well defined.

We prove the following generalization of a well-known result for the Yang—Baxter equation.

Proposition 4.1. *Let $R : X^n \to X^n$ be a solution of the n-simplex equation.*

1. *If R is invertible then the inverse mapping R^{-1} is a solution of the n-simplex equation too.*
2. *If $\varphi_1, \dots, \varphi_n$ are pairwise commuting mappings on X then the mapping $R : X^n \to X^n$ with*

$$R(x_1, \dots, x_n) = (\varphi_1(x_1), \dots, \varphi_n(x_n)),$$

is a solution of the n-simplex equation.

3. If $\varphi \in Sym(X)$ is a bijection on X then the composition

$$\varphi^{\times n}\, R\, (\varphi^{-1})^{\times n}$$

is a solution of the n-simplex equation.

Proof. (1) We consider the inverses of the mappings on the left-hand and right-hand sides of the n-simplex equation. We find that R^{-1} is a solution of the n-simplex equation.

(2) It is easy to see that the mappings of the form $R_{\bar{i}}$ obtained from R commute pairwise. We may permute mappings on the left-hand side of the n-simplex equation and obtain the expression on the right-hand side.

(3) Let $\widetilde{R} = \varphi^{\times n}\, R\, (\varphi^{-1})^{\times n}$. Then we have

$$\widetilde{R}_{\bar{k}} = \varphi^{\times n}_{\bar{k}}\, R_{\bar{k}}\, (\varphi^{-1})^{\times n}_{\bar{k}} = \varphi^{\times N}\, R_{\bar{k}}\, (\varphi^{-1})^{\times N}.$$

The equality

$$\widetilde{R}_{\bar{1}}\widetilde{R}_{\bar{2}} \cdots \widetilde{R}_{\overline{n+1}} = \widetilde{R}_{\overline{n+1}} \cdots \widetilde{R}_{\bar{2}}\widetilde{R}_{\bar{1}}$$

holds if and only if

$$\begin{aligned}&\left(\varphi^{\times N}\, R_{\bar{1}}\, (\varphi^{-1})^{\times N}\right)\, \left(\varphi^{\times N}\, R_{\bar{2}}\, (\varphi^{-1})^{\times N}\right) \cdots \left(\varphi^{\times N}\, R_{\overline{n+1}}\, (\varphi^{-1})^{\times N}\right)\\ &\qquad = \left(\varphi^{\times N}\, R_{\overline{n+1}}\, (\varphi^{-1})^{\times N}\right) \cdots \left(\varphi^{\times N}\, R_{\bar{2}}\, (\varphi^{-1})^{\times N}\right)\, \left(\varphi^{\times N}\, R_{\bar{1}}\, (\varphi^{-1})^{\times N}\right).\end{aligned}$$

We cancel and obtain

$$\varphi^{\times N} R_{\bar{1}} R_{\bar{2}} \cdots R_{\overline{n+1}} (\varphi^{-1})^{\times N} = \varphi^{\times N} R_{\overline{n+1}} \cdots R_{\bar{2}} R_{\bar{1}}\, (\varphi^{-1})^{\times N}.$$

Since φ^{N} is a bijective mapping, the obtained equation is equivalent to the initial one. □

We present a generalization of [24, Proposition 2.2].

Proposition 4.2. *If R is a solution of the n-simplex equation and $\varphi \in Sym(X)$ is a bijective mapping with*

$$(\varphi^{-1})^{\times n}\, R\, \varphi^{\times n} = R$$

then the mapping

$$\tilde{R} = (\varphi \times \mathrm{id} \times \varphi \times \ldots)\, R\, (\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots)$$

determines a solution of the n-simplex equation, where $(\varphi \times \mathrm{id} \times \varphi \times \ldots)$ acts as φ on the odd factors and as the identity mapping on the even factors, while $(\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots)$ acts as φ^{-1} on the even factors and as the identity mapping on the odd factors.

Proof. Notice that

$$(\varphi \times \mathrm{id} \times \varphi \times \ldots) = (\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots)\, \varphi^{\times n},$$
$$(\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots) = (\varphi^{-1})^{\times n}\, (\varphi \times \mathrm{id} \times \varphi \times \ldots),$$

and

$$\begin{aligned}R_A := \tilde{R} &= (\varphi \times \mathrm{id} \times \varphi \times \ldots) R (\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots)\\ &= (\varphi \times \mathrm{id} \times \varphi \times \ldots)\left((\varphi^{-1})^{\times n} R \varphi^{\times n}\right)(\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots)\\ &= (\mathrm{id} \times \varphi^{-1} \times \mathrm{id} \times \ldots) R (\varphi \times \mathrm{id} \times \varphi \times \ldots) =: R_B.\end{aligned}$$

In the equation, each index occurs exactly two times. The jth entry of $\bar{i}$ is the ith entry of $\overline{j+1}$. We replace each mapping $R_{\bar{i}}$, where i is even, with $(R_A)_{\bar{i}}$ and replace each mapping $R_{\bar{i}}$, where i is odd, with $(R_B)_{\bar{i}}$. For all i and $j \geqslant i$, the action on each component between $R_{\bar{i}}$ and $R_{\overline{j+1}}$ is the identity mapping. Indeed, if i and j are of the same parity then this action is $\mathrm{id} \circ \mathrm{id}$; if the parities of i and j are different then the corresponding action is either $\varphi \circ \varphi^{-1}$ or $\varphi^{-1} \circ \varphi$. Since each index occurs exactly two times, the equation

$$(R_B)_{\overline{1}}(R_A)_{\overline{2}} \dots (\tilde{R})_{\overline{n+1}} = (\tilde{R})_{\overline{n+1}} \dots (R_A)_{\overline{2}}(R_B)_{\overline{1}}$$

is reduced to the equation

$$\Phi R_{\overline{1}} R_{\overline{2}} ... R_{\overline{n+1}} \Phi = \Phi R_{\overline{n+1}} ... R_{\overline{2}} R_{\overline{1}} \Phi,$$

where Φ denotes the component-wise action by either φ, or φ^{-1}, or id. □

Let (X, R) be an elementary solution of the Yang—Baxter equation, let $R(x, y) = (x, y * x)$, and let $P(x, y) = (y, x)$ for all $x, y \in X$. It is not difficult to verify that

$$PRP(x, y) = (x * y, y)$$

is an elementary solution of the Yang—Baxter equation.

The proof of the following assertion is elementary. We present the arguments because they will be generalized later to the case of the n-simplex equation for an arbitrary n.

Lemma 4.3. *If $R(x, y) = (x \cdot y, y * x)$ is a solution of the Yang—Baxter equation then $PRP(x, y) = (x * y, y \cdot x)$ is a solution of the Yang—Baxter equation too.*

Proof. It is not difficult to see that

$$R_{13} = P_{23} R_{12} P_{23}, \quad R_{23} = P_{12} P_{23} R_{12} P_{23} P_{12},$$

where $P_{ij} : X^3 \to X^3$ permutes the ith and jth components. Then the Yang—Baxter equation assumes the form

$$R_{12} \cdot R_{12}^{P_{23}} \cdot R_{12}^{P_{23}P_{12}} = R_{12}^{P_{23}P_{12}} \cdot R_{12}^{P_{23}} \cdot R_{12}.$$

We denote $\widetilde{R} = PRP$. Then $R = P\widetilde{R}P$ and from the Yang—Baxter equation it follows that

$$\widetilde{R}_{12}^{P_{12}} \cdot \widetilde{R}_{12}^{P_{12}P_{23}} \cdot \widetilde{R}_{12}^{P_{12}P_{23}P_{12}} = \widetilde{R}_{12}^{P_{12}P_{23}P_{12}} \cdot \widetilde{R}_{12}^{P_{12}P_{23}} \cdot \widetilde{R}_{12}^{P_{12}}.$$

We conjugate on both sides of the equation by $P_{12}P_{23}P_{12} = P_{13}$. We obtain

$$\widetilde{R}_{12}^{P_{23}P_{12}} \cdot \widetilde{R}_{12}^{P_{23}} \cdot \widetilde{R}_{12} = \widetilde{R}_{12} \cdot \widetilde{R}_{12}^{P_{23}} \cdot \widetilde{R}_{12}^{P_{23}P_{12}}.$$

This equation is equivalent to the equation

$$\widetilde{R}_{12}\widetilde{R}_{12}\widetilde{R}_{12} = \widetilde{R}_{12}\widetilde{R}_{12}\widetilde{R}_{12}.$$

We conclude that $\widetilde{R} = PRP$ is a solution of the Yang—Baxter equation. □

We generalize this lemma. We consider a solution (X, R) of the n-simplex equation and introduce a permutation $P \in Sym(X^n)$. We put

$$P = P_{1,n} P_{2,n-1} \dots P_{[n/2]+1, n-[n/2]},$$

where $[n/2]$ denotes the integer part of $n/2$ and, for $n = 2m + 1$, we assume that the ultimate permutation $P_{[n/2]+1,n-[n/2]} = P_{m+1,m+1}$ is the identity mapping. Notice that $P^{-1} = P$.

Proposition 4.4. *If $R : X^n \to X^n$ is a solution of the n-simplex equation then $\widetilde{R} = PRP$ is a solution of the n-simplex equation too.*

Proof. We rewrite the n-simplex equation

$$R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}} = R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}}$$

in the form

$$R_{\overline{1}} R_{\overline{1}}^{Q_1} \cdots R_{\overline{1}}^{Q_n} = R_{\overline{1}}^{Q_n} \cdots R_{\overline{1}}^{Q_1} R_{\overline{1}},$$

where $Q_1, \dots, Q_n \in Sym(X^n)$ are permutations. Since $R = P\widetilde{R}P$, we have

$$\widetilde{R}_{\overline{1}}^{P} \widetilde{R}_{\overline{1}}^{PQ_1} \cdots \widetilde{R}_{\overline{1}}^{PQ_n} = \widetilde{R}_{\overline{1}}^{PQ_n} \cdots \widetilde{R}_{\overline{1}}^{PQ_1} \widetilde{R}_{\overline{1}}^{P}.$$

We conjugate on both sides of the equality by $Q_n^{-1}P$. We obtain

$$\widetilde{R}_{\bar{1}}^{PQ_n^{-1}P}\widetilde{R}_{\bar{1}}^{PQ_1Q_n^{-1}P}\cdots\widetilde{R}_{\bar{1}}^{PQ_{n-1}Q_n^{-1}P}\widetilde{R}_{\bar{1}}=\widetilde{R}_{\bar{1}}\widetilde{R}_{\bar{1}}^{PQ_{n-1}Q_n^{-1}P}\cdots\widetilde{R}_{\bar{1}}^{PQ_1Q_n^{-1}P}\;\widetilde{R}_{\bar{1}}^{PQ_n^{-1}P}.$$

We apply the equalities

$$PQ_n^{-1}P=Q_n,\;\;PQ_1Q_n^{-1}P=Q_{n-1},\ldots,PQ_{n-1}Q_n^{-1}P=Q_1.$$

We obtain

$$\widetilde{R}_{\bar{1}}^{Q_n}\widetilde{R}_{\bar{1}}^{Q_{n-1}}\cdots\widetilde{R}_{\bar{1}}^{Q_1}\widetilde{R}_{\bar{1}}=\widetilde{R}_{\bar{1}}\widetilde{R}_{\bar{1}}^{Q_1}\cdots\widetilde{R}_{\bar{1}}^{Q_{n-1}}\;\widetilde{R}_{\bar{1}}^{Q_n}.$$

We conclude that $\widetilde{R}$ is a solution of the n-simplex equation. □

Question 4.5. As is known, if R is a solution of the Yang–Baxter equation then $S=PR$ satisfies the braid relation $S_{12}S_{23}S_{12}=S_{23}S_{12}S_{23}$. What corresponds to this relation in the case of the n-simplex equation?

4.1. *Composition of solutions.* Two operations are defined on the set of linear solutions of the Yang–Baxter equation (see, for example, [30]); namely, the tensor product and the direct sum. If $(V_1,R^{(1)})$ and $(V_2,R^{(2)})$ are solutions of the Yang–Baxter equation then the tensor product $(V_1\otimes V_2,R^{(1)}\otimes R^{(2)})$ and the direct sum $(V_1\times V_2,R^{(1)}+R^{(2)})$ are solutions of the Yang–Baxter equation too. The mapping $R^{(1)}+R^{(2)}$ is defined on $(V_1\times V_2)\otimes(V_1\times V_2)$ by the rule

$$\begin{aligned}(R^{(1)}+R^{(2)})(e_i\otimes e_j)&=R^{(1)}(e_i\otimes e_j),\\(R^{(1)}+R^{(2)})(e_i\otimes f_q)&=e_i\otimes f_q,\\(R^{(1)}+R^{(2)})(f_p\otimes e_j)&=f_p\otimes e_j,\\(R^{(1)}+R^{(2)})(f_p\otimes f_q)&=R^{(2)}(f_p\otimes f_q),\end{aligned}$$

where $\{e_\alpha\}$ is a basis of V_1 and $\{f_\beta\}$ is a basis of V_2.

Assume that (X,A) and (Y,B) are set-theoretical solutions of the n-simplex and m-simplex equation respectively. In the present subsection, we address the question on construction of new solutions.

If $m=n$ then it is easy to construct the direct product of solutions. We have

$$A\times B:(X\times Y)^n\to(X\times Y)^n,$$
$$A\times B\left((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\right)=\left((f_1(\bar{x}),g_1(\bar{y})),(f_2(\bar{x}),g_2(\bar{y})),\ldots,(f_n(\bar{x}),g_n(\bar{y}))\right),$$

where

$$A(\bar{x})=(f_1(\bar{x}),f_2(\bar{x}),\ldots,f_n(\bar{x})),\quad B(\bar{y})=(g_1(\bar{y}),g_2(\bar{y}),\ldots,g_n(\bar{y}))$$

for $\bar{x}=(x_1,x_2,\ldots,x_n)$ and $\bar{y}=(y_1,y_2,\ldots,y_n)$. The mapping $A\times B$ is a solution of the n-simplex equation.

Definition 4.6. Let $A\colon X^{n+k}\to X^{n+k}$ and $B\colon X^{k+m}\to X^{k+m}$ be mappings of the form

$$A(\bar{x},\bar{y})=\left(f_1(\bar{x},\bar{y}),\ldots,f_n(\bar{x},\bar{y}),h_1(\bar{y}),\ldots,h_k(\bar{y})\right),\quad\bar{x}\in X^n,\ \bar{y}\in X^k,$$
$$B(\bar{y},\bar{z})=\left(h_1(\bar{y}),\ldots,h_k(\bar{y}),g_1(\bar{y},\bar{z}),\ldots,g_m(\bar{y},\bar{z})\right),\quad\bar{z}\in X^m.$$

Then the mapping $A\,\#_kB\colon X^{n+k+m}\to X^{n+k+m}$ defined by the rule

$$A\,\#_kB(\bar{x},\bar{y},\bar{z}):=\left(f_1(\bar{x},\bar{y}),\ldots,f_n(\bar{x},\bar{y}),h_1(\bar{y}),\ldots,h_k(\bar{y}),g_1(\bar{y},\bar{z}),\ldots,g_m(\bar{y},\bar{z})\right)$$

is called the k-*amalgam* of A and B.

Theorem 4.7. *We consider positive integers n and m and a nonnegative integer k. Let $A\colon X^{n+k}\to X^{n+k}$ be a solution of the $(n+k)$-simplex equation and let $B\colon X^{k+m}\to X^{k+m}$ be a solution of the $(m+k)$-simplex equation such that*

$$A(\bar{x},\bar{y})=\left(f_1(\bar{x},\bar{y}),\ldots,f_n(\bar{x},\bar{y}),h_1(\bar{y}),\ldots,h_k(\bar{y})\right),\quad\bar{x}\in X^n,\ \bar{y}\in X^k,$$

$$B(\bar{y},\bar{z}) = (h_1(\bar{y}),\dots,h_k(\bar{y}),g_1(\bar{y},\bar{z}),\dots,g_m(\bar{y},\bar{z}))\,, \quad \bar{z}\in X^m.$$

Then the k-amalgam $A\,\#_k B : X^{n+k+m}\to X^{n+k+m}$ of A and B defined by the rule

$$A\,\#_k B(\bar{x},\bar{y},\bar{z}) = (f_1(\bar{x},\bar{y}),\dots,f_n(\bar{x},\bar{y}),h_1(\bar{y}),\dots,h_k(\bar{y}),g_1(\bar{y},\bar{z}),\dots,g_m(\bar{y},\bar{z}))$$

is a solution of the $(n+k+m)$-simplex equation if and only if, for all i and j with $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$ and all elements $a_{\alpha,\beta}\in X$ with $\alpha\in\{1,\dots,n+k\}$ and $\beta\in\{1,\dots,k+m\}$,

$$f_i\left(\begin{array}{lllllllll}
g_j(& a_{1,1}, & a_{1,2}, & \cdots & a_{1,k-1}, & a_{1,k}, & \cdots & a_{1,k+m} &),\\
g_j(& a_{2,1}, & a_{2,2}, & \cdots & a_{2,k-1}, & a_{2,k}, & \cdots & a_{2,k+m} &),\\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
g_j(& a_{n+1,1}, & a_{n+1,2}, & \cdots & a_{n+1,k-1}, & a_{n+1,k}, & \cdots & a_{n+1,k+m} &),\\
g_j(& b^{n+1,1}, & a_{n+2,2}, & \cdots & a_{n+2,k-1}, & a_{n+2,k}, & \cdots & a_{n+2,k+m} &),\\
g_j(& b^{n+1,2}, & b^{n+2,2}, & \cdots & a_{n+3,k-1}, & a_{n+3,k}, & \cdots & a_{n+3,k+m} &),\\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
g_j(& b^{n+1,k-1}, & b^{n+2,k-1}, & \cdots & b^{n+k-1,k-1} & a_{n+k,k}, & \cdots & a_{n+k,k+m} &)
\end{array}\right)$$

$$\|$$

$$g_j\left(\begin{array}{lllllllll}
f_i(& a_{1,1}, & a_{2,1}, & \cdots & a_{n+1,1}, & b_{2,2}, & b_{3,2}, & \cdots & b_{k,2} &),\\
f_i(& a_{1,2}, & a_{2,2}, & \cdots & a_{n+1,2}, & a_{n+2,2}, & b_{3,3}, & \cdots & b_{k,3} &),\\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
f_i(& a_{1,k-1}, & a_{2,k-1}, & \cdots & a_{n+1,k-1}, & a_{n+2,k-1}, & a_{n+3,k-1}, & \cdots & b_{k,k} &),\\
f_i(& a_{1,k}, & a_{2,k}, & \cdots & a_{n+1,k}, & a_{n+2,k}, & a_{n+3,k}, & \cdots & a_{n+k,k} &),\\
 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
f_i(& a_{1,k+m}, & a_{2,k+m}, & \cdots & a_{n+1,k+m} & a_{n+2,k+m}, & a_{n+3,k+m}, & \cdots & a_{n+k,k+m} &)
\end{array}\right),$$

where

$$\begin{aligned}
b^{i,j} &:= h_j(b^{n+1,i-n-1},\dots,b^{i-1,i-n-1},a_{i,j},\dots,a_{i,k}),\\
b_{i,j} &:= h_j(a_{n+1,i},\dots,a_{n+i,i},b_{i+1,i},\dots,b_{k,i}).
\end{aligned}$$

Proof. The assertion is clear from inspection of the matrix MI_{n+k+m} of the $(n+k+m)$-simplex equation. The $((n+k)\times(n+k+1))$-submatrix at the left upper corner and the $((k+m)\times(k+m+1))$-submatrix at the right lower corner are independent of the remaining entries of the matrix and correspond to the $(n+k)$-simplex and $(k+m)$-simplex equations.

Since each function h_i is independent of the first n and last m variables, the common entries of these submatrices are independent of the remaining entries of the matrix. Therefore, the $(n+k+m)$-simplex equation splits into the $(n+k)$-simplex and $(k+m)$-simplex equations on indices in the corresponding submatrices. The remaining submatrices at the left lower and right upper corners consist of nm distinct entries and are transposes to each other. The explicit expression of the $(n+k+m)$-simplex equation on indices in these matrices leads to the above condition. □

We illustrate the construction of the k-amalgam with a series of examples.

Example 4.8. Let X be a set and let $A,B : X^2\to X^2$ be solutions of the Yang–Baxter equation such that

$$A(x,y) = (f_1(x,y),f_2(x,y)), \quad B(z,t) = (g_1(z,t),g_2(z,t)), \quad x,y,z,t\in X.$$

By Lemma 2.1, we have

$$\begin{aligned} f_1\left(f_1(x,y),z\right) &= f_1\left(f_1(x,f_2(y,z)),f_1(y,z)\right), \\ f_1\left(f_2(x,y),f_2(f_1(x,y),z)\right) &= f_2\left(f_1(x,f_2(y,z)),f_1(y,z)\right), \\ f_2\left(f_2(x,y),f_2(f_1(x,y),z)\right) &= f_2\left(x,f_2(y,z)\right) \end{aligned}$$

and

$$\begin{aligned} g_1\left(g_1(x,y),z\right) &= g_1\left(g_1(x,g_2(y,z)),g_1(y,z)\right), \\ g_1\left(g_2(x,y),g_2(g_1(x,y),z)\right) &= g_2\left(g_1(x,g_2(y,z)),g_1(y,z)\right), \\ g_2\left(g_2(x,y),g_2(g_1(x,y),z)\right) &= g_2\left(x,g_2(y,z)\right) \end{aligned}$$

for all $x,y,z,t \in X$. We put

$$R(x,y,z,t) = (f_1(x,y), f_2(x,y), g_1(z,t), g_2(z,t)), \quad x,y,z,t \in X.$$

We describe conditions guaranteeing that the mapping R satisfies the 4-simplex equation

$$R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10} = R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}.$$

Straightforward calculation shows that

$$\begin{aligned} &R_{479,10}R_{368,10}R_{2589}R_{1567}R_{1234}(x,y,z,t,p,q,r,s,u,v) \\ &\quad = (f_1(f_1(x,y),p), f_1(f_2(x,y),f_2(f_1(x,y),p)), f_1(g_1(z,t),g_1(q,r)), f_1(g_2(z,t),g_2(q,r)), \\ &\quad\quad f_2(f_2(x,y),f_2(f_1(x,y),p)), f_2(g_1(z,t),g_1(q,r)), f_2(g_2(z,t),g_2(q,r)), g_1(g_1(s,u),v), \\ &\quad\quad g_1(g_2(s,u),g_2(g_1(s,u),v)), g_2(g_2(s,u),g_2(g_1(s,u),v))) \end{aligned}$$

and

$$\begin{aligned} &R_{1234}R_{1567}R_{2589}R_{368,10}R_{479,10}(x,y,z,t,p,q,r,s,u,v) \\ &\quad = (f_1(f_1(x,f_2(y,p)),f_1(y,p)), f_2(f_1(x,f_2(y,p)),f_1(y,p)), g_1(f_1(z,q),f_1(t,r)), \\ &\quad\quad g_2(f_1(z,q),f_1(t,r)), f_2(x,f_2(y,p)), g_1(f_2(z,q),f_2(t,r)), g_2(f_2(z,q),f_2(t,r)), \\ &\quad\quad g_1(g_1(s,g_2(u,v)),g_1(u,v)), g_2(g_1(s,g_2(u,v)),g_1(u,v)), g_2(s,g_2(u,v))). \end{aligned}$$

We equate the expressions on the left-hand and right-hand sides. We obtain a system of ten equations. The first, second, and fifth equations together mean that A is a solution of the Yang–Baxter equation. The last three equations mean that B is a solution of the Yang–Baxter equation. Therefore, the mapping R is a solution of the 4-simplex equation if and only if

$$\begin{aligned} f_1(g_1(z,t),g_1(q,r)) &= g_1(f_1(z,q),f_1(t,r)), \\ f_1(g_2(z,t),g_2(q,r)) &= g_2(f_1(z,q),f_1(t,r)), \\ f_2(g_1(z,t),g_1(q,r)) &= g_2(f_2(z,q),f_2(t,r)), \\ f_2(g_2(z,t),g_2(q,r)) &= g_2(f_2(z,q),f_2(t,r)) \end{aligned}$$

for all $z,t,q,r \in X$. Notice that these equalities coincide with the conditions on f_i and g_j from Theorem 4.7.

Example 4.9. Let X be a set and let $A, B : X^2 \to X^2$ be solutions of the Yang–Baxter equation such that

$$A(x,y) = (f(x,y), h(y)), \quad B(y,z) = (h(y), g(y,z)), \quad x,y,z \in X.$$

By Lemma 2.1, we have

$$\begin{aligned} &f\left(f(x,y),z\right) = f\left(f(x,h(z)),f(y,z)\right), && f(h(y),h(z)) = h(f(y,z)), \\ &g\left(y,g(z,t)\right) = g\left(g(y,z),g(h(y),t)\right), && g(h(y),h(z)) = h(g(y,z)) \end{aligned}$$

for all $x,y,z,t \in X$. We consider the mapping

$$R(x,y,z) = (f(x,y), h(y), g(y,z))$$

from X^3 to X^3. We have

$$R_{123}R_{145}R_{246}R_{356}(x,y,z,t,p,q)$$
$$= \big(f(f(x,h(t)),f(y,t)),h(f(y,t)),g(f(y,t),f(z,p)),h^2(t),g(h(t),h(p)),g(t,g(p,q))\big)$$

and

$$R_{356}R_{246}R_{145}R_{123}(x,y,z,t,p,q)$$
$$= \big(f(f(x,y),t),f(h(y),h(t)),f(g(y,z),g(t,p)),h^2(t),h(g(t,p)),g(g(t,p),g(h(t),q))\big).$$

We conclude that R is a solution of the tetrahedron equation if and only if, for all $x,y,z,t,p,q \in X$, we have

$$\begin{aligned} f\left(f(x,h(t)),f(y,t)\right) &= f\left(f(x,y),t\right), \\ h(f(y,t)) &= f(h(y),h(t)), \\ g\left(f(y,t),f(z,p)\right) &= f\left(g(y,z),g(t,p)\right), \\ g(h(t),h(p)) &= h(g(t,p)), \\ g\left(t,g(p,q)\right) &= g\left(g(t,p),g(h(t),q)\right). \end{aligned}$$

If A and B are solutions of the Yang–Baxter equation then the first, second, fourth, and fifth equalities hold. We find that R is a solution of the tetrahedron equation if and only if, for all $y,z,t,p \in X$, we have

$$g\left(f(y,t),f(z,p)\right) = f\left(g(y,z),g(t,p)\right).$$

In this case, we obtain $n=m=k=1$ and, by Theorem 4.7, the mapping $A\,\#_1 B : X^3 \to X^3$, where

$$A\,\#_1 B(x,y,z) = (f(x,y),h(y),g(y,z)),$$

is a solution of the tetrahedron equation if and only if the following equality holds:

$$g\left(f(a_{11},a_{21}),f(a_{12},a_{22})\right) = f\left(g(a_{11},a_{12}),g(a_{21},a_{22})\right).$$

Remark 4.10. Notice that, for $k \in \{0,1\}$, the elements $b^{i,j}$ and $b_{i,j}$ do not occur in the conditions of Theorem 4.7. It is easy to see that, for $k=0$, if $A\#_0 B$ is a solution of the n-simplex equation then $B\#_0 A$ is a solution of the n-simplex equation too.

4.2. *Simple solutions.* In the present subsection, we define a certain class of solutions of the n-simplex equation.

Definition 4.11. A solution $T : X^n \to X^n$ of the n-simplex equation is said to be *simple* if

$$T(x_1,\dots,x_n) = (x_{s(1)},\dots,x_{s(n)}),$$

where $s : \{1,\dots,n\} \to \{1,\dots,n\}$ is an arbitrary mapping.

Example 4.12. Let X be an arbitrary set.

(1) The identity mapping $\mathrm{id} : X \to X$ is a simple solution of the 1-simplex equation.

(2) The mappings $P, Pr_1^2, Pr_2^2 : X^2 \to X^2$ defined by the rules

$$\begin{aligned} P:&\quad (x,y) \mapsto (y,x), \\ Pr_1^2:&\quad (x,y) \mapsto (x,x), \\ Pr_2^2:&\quad (x,y) \mapsto (y,y) \end{aligned}$$

for all $x,y,z \in X$ are simple solutions of the 2-simplex equation.

(3) The mapping $Pr_2^3 : X^3 \to X^3$ defined by the rule

$$Pr_2^3 : (x,y,z) \mapsto (y,y,y)$$

for all $x,y,z \in X$ is a simple solution of the 3-simplex equation.

It is obvious that the product of simple solutions is a simple solution too.

The following assertion is immediate from Theorem 4.7.

Proposition 4.13. *Let $R : X^n \to X^n$ be a solution of the n-simplex equation and let $A \in \{\mathrm{id}_X, P, Pr_1^2, Pr_2^2, Pr_2^3\}$. Then $R\#_0 A$ and $A\#_0 R$ are solutions of the $(n+m)$-simplex equation.*

Definition 4.14. A solution R is said to be *indecomposable* if $R = A\#_0 B$ for no solutions A and B.

Question 4.15. (1) Is there a natural n such that the n-simplex equation admits simple indecomposable solutions that do not belong to $\{\mathrm{id}_X, P, Pr_1^2, Pr_2^2, Pr_2^3\}$?

(2) As is known, the permutation P_{12} is a solution of the Yang–Baxter equation. Describe all $n > 2$ such that permutations without fixed points are solutions of the n-simplex equation.

Proposition 4.4 allows us to find permutations that give rise to solutions of the n-simplex equation.

4.2. *Linear and affine solutions.* Linear and affine solutions of the Yang–Baxter and tetrahedron equations were studied in [9, 10, 21, 25].

Theorem 4.7 allows us to prove the following assertion.

Proposition 4.16. *Let A and B be linear solutions of the n-simplex and m-simplex equation respectively. Then their 0-amalgam $A\#_0 B$ is a linear solution of the $(n+m)$-simplex equation. If the 1-amalgam $A\#_1 B$ is defined for A and B than it forms a solution of the $(n+m-1)$-simplex equation.*

Proof. Let $f\colon X^n \to X$ be a linear mapping. We rewrite it in the matrix form

$$f(\bar{x}) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = \bar{x}^T [f] = [f]^T \bar{x},$$

where $[f]^T = (a_1, a_2, \ldots, a_n)$ is the tuple of coefficients, $\bar{x}^T = (x_1, x_2, \ldots, x_n)$ is the tuple of variables, and $\cdot^T$ denotes the transposition operation. We rewrite the conditions of Theorem 4.7 for $k \in \{0, 1\}$ as follows:

$$[f]^T M [g] = [g]^T M^T [f].$$

Here f denotes a component of A, g denotes a component of B, and M is an $(n \times m)$-matrix.

This means that, for all linear solutions A and B, the mappings $A\ \#_k B$ with $k \in \{0, 1\}$ are solutions of the corresponding n-simplex equation. □

4.3. *Construction of rational solutions from linear solutions.* We consider a linear solution $(\mathbb{R}, R)$ of the Yang–Baxter equation, i.e., let

$$R(x, y) = (\alpha_1 x, (1 - \alpha_1\alpha_2)x + \alpha_2 y), \text{ where } \alpha_1, \alpha_2 \in \mathbb{R}.$$

Proposition 4.16 provides us with a solution $R : \mathbb{R}^n \to \mathbb{R}^n$ of the n-simplex equation of the form

$$x_i \mapsto \begin{cases} \alpha_i x_i & \text{if } i \text{ is odd,} \\ (1 - \alpha_{i-1}\alpha_i)x_{i-1} + \alpha_i x_i + (1 - \alpha_i\alpha_{i+1})x_{i+1} & \text{if } i \text{ is even} \\ (1 - \alpha_{i-1}\alpha_i)x_{n-1} + \alpha_i x_n & \text{if } i = n \text{ is even,} \end{cases}$$

where $\alpha_i \in \mathbb{R}$.

We consider a rational function $f : \mathbb{R} \to \mathbb{R}$ such that f^{-1} is a rational function too. For example, we may consider a linear-fractional transform with

$$f(x) = \frac{ax + b}{cx + d}, \quad f^{-1}(x) = \frac{dx - b}{a - cx}.$$

We conjugate the solution by f. We obtain a rational solution $(f^{-1})^{\times n} R f^{\times n}$. For a linear-fractional transform f, the solution assumes the form

$$x_i \mapsto \begin{cases} \dfrac{(\alpha_i ad - bc)x_i + (\alpha_i - 1)bd}{ca(1 - \alpha_i)x_i + (ad - \alpha_i bc)} & \text{if } i \text{ is odd,} \\ \dfrac{d\left(\beta_i \frac{ax_{i-1}+b}{cx_{i-1}+d} + \alpha_i \frac{ax_i+b}{cx_i+d} + \beta_{i+1}\frac{ax_{i+1}+b}{cx_{i+1}+d}\right) - b}{a - c\left(\beta_i \frac{ax_{i-1}+b}{cx_{i-1}+d} + \alpha_i \frac{ax_i+b}{cx_i+d} + \beta_{i+1}\frac{ax_{i+1}+b}{cx_{i+1}+d}\right)} & \text{if } i \text{ is even,} \\ \dfrac{d\left(\beta_n \frac{ax_{n-1}+b}{cx_{n-1}+d} + \alpha_n \frac{ax_n+b}{cx_n+d}\right) - b}{a - c\left(\beta_n \frac{ax_{n-1}+b}{cx_{n-1}+d} + \alpha_n \frac{ax_n+b}{cx_n+d}\right)} & \text{if } i = n \text{ is even,} \end{cases}$$

where $\beta_i = 1 - \alpha_{i-1}\alpha_i$ for $i \in \{2, 3, \ldots, n\}$. This construction allows us to construct a set of new rational solutions of the n-simplex equation. It is interesting to know which rational solutions can be obtained from linear solutions by conjugation.

Example 4.17. We consider

$$\alpha_1 = \alpha_3 = 1,\ \alpha_2 = \alpha_4 = 0.$$

Then we have

$$\beta_2 = \beta_3 = \beta_4 = 1.$$

Let

$$a = c = 1,\ \ b = 0,\ \ d = -1.$$

We obtain a solution of the 4-simplex equation

$$(x_1, x_2, x_3, x_4) \mapsto \left(x_1, \frac{x_1 + x_3 - 2x_1x_3}{1 - x_1x_3}, x_3, x_3\right).$$

Example 4.18. Let R be the solution of the 4-simplex equation defined by the rule

$$R(x_1, x_2, x_3, x_4) = (x_2 - x_4, x_1 + x_3, x_3, x_4).$$

For f we take the mapping $x \mapsto \dfrac{x}{x-1}$. We obtain a solution of the 4-simplex equation

$$(f^{-1})^{\times n} R f^{\times n}(x_1, x_2, x_3, x_4) = \left(\frac{x_2 - x_4}{1 + x_2x_4}, \frac{x_1 + x_3 - 2x_1x_3}{1 - x_1x_3}, x_3, x_4\right).$$

Question 4.19. Is it possible to obtain the electric solution by conjugating a linear solution?

4.4. *From solutions of the n-simplex equation to solutions of the $(n-1)$-simplex equation.* The construction of the k-amalgam defined above allows us to construct a new solution of larger dimension form a pair of solutions. We turn to an opposite problem, i.e., is it possible, for a solution (X, R) of the n-simplex equation, to construct a solution of the $(n-1)$-simplex equation?

Proposition 4.20. *Let $R : X^n \to X^n$ for $n \geq 3$, let R be a solution of the n-simplex equation, and let there exist $x_0 \in X$ such that $R(x_0, \dots, x_0) = (x_0, \dots, x_0)$. Then*

$$R^r(x_1, \dots, x_{n-1}) := R(x_0, x_1, \dots, x_{n-1}), \quad R^l(x_1, \dots, x_{n-1}) := R(x_1, \dots, x_{n-1}, x_0)$$

are solutions of the $(n-1)$-simplex equation.

Proof. If R is a solution of the n-simplex equation then the mappings $R_{\bar{i}}$ with $1 \leq i \leq n+1$ act on X^N, where $N = n(n+1)/2$. We consider a set $X_{x_0} \subset X^N$ such that the first n coordinates are equal to x_0, i.e., we have

$$X_{x_0} = \{(\underbrace{x_0, \dots, x_0}_{n}, x_{n+1}, x_{n+2}, \dots, x_N) \in X^N\}.$$

We restrict the solution of the n-simplex equation to X_{x_0}. On the right-hand side of the equation, we apply $R_{\bar{1}}$:

$$R_{\bar{1}}(\underbrace{x_0, \dots, x_0}_{n}, x_{n+1}, x_{n+2}, \dots, x_N) = R(x_0, \dots, x_0) \times \mathrm{id}^{N-n}(x_{n+1}, x_{n+2}, \dots, x_N)$$
$$= (\underbrace{x_0, \dots, x_0}_{n}, x_{n+1}, x_{n+2}, \dots, x_N).$$

We find that $R_{\bar{1}}$ acts as the identity mapping on X_{x_0}. On the set X_{x_0}, the n-simplex equation assumes the form

$$R_{\bar{1}} R_{\bar{2}} \dots R_{\overline{n+1}} = R_{\overline{n+1}} \dots R_{\bar{2}}.$$

The mappings on both sides of the $(n-1)$-simplex equation act on $X^{N'}$, where $N' = N - n$. We remove the first n coordinates and consider the above equation restricted to the last N' coordinated. We obtain the $(n-1)$-simplex equation after the shift of all indices by $-n$.

We show that R^l is a solution of the $(n-1)$-simplex equation. We consider the set X^N, where the coordinates with numbers in $\overline{n+1}$ are equal to x_0. We notice that $R_{\overline{n+1}}$ acts on this set as the identity mapping. □

5. TROPICALIZATION

5.1. *Tropicalization of rational solutions.* By tropicalization we mean a certain mapping from the set of rational functions to the set of piecewise-linear functions. Dynnikov [16] used tropicalization for studying representations of the braid group B_n in the permutation group $Sym(\mathbb{Z}^{2n})$. In particular, he found nondegenerate solutions of the Yang—Baxter equation on $\mathbb{Z}^2$.

In the present section, we establish a connection between rational and piecewise-linear solutions of the n-simplex equation for an arbitrary n.

In the formulation of the main result, we need a series of definition and the following notation. Let $\mathbb{R}(x_1, x_2, \ldots, x_n)$ denote the field of rational functions over the set $\mathbb{R}$ of reals. A solution $(\mathbb{R}, R)$ of the n-simplex equation, where

$$R(x_1, x_2, \ldots, x_n) = (r_1, r_2, \ldots, r_n), \quad r_i \in \mathbb{R}(x_1, x_2, \ldots, x_n),$$

is called a *rational solution*.

Let I_n denote the subset of nonzero fractions of the form $r = f/g \in \mathbb{R}(x_1, x_2, \ldots, x_n)$ such that each nonzero coefficient of f is equal to 1, the constant term of f vanishes and either $g = 1$ or each nonzero coefficient of g is equal to 1 while the constant term of g vanishes. A rational solution

$$R(x_1, x_2, \ldots, x_n) = (r_1, r_2, \ldots, r_n), \quad r_i \in \mathbb{R}(x_1, x_2, \ldots, x_n),$$

of the n-simplex equation is said to be *I-rational* if all components r_i belong to I_n.

Example 5.1. It is easy to see that the electric solution

$$R_E(x, y, z) = \left(\frac{xy}{x + z + xyz},\ x + z + xyz,\ \frac{yz}{x + z + xyz} \right)$$

of the tetrahedron equation which arises in the electric network theory and is well-known as the star-triangle transform is an I-rational solution.

The mapping

$$R_e(x, y, z) = \left(\frac{xy}{x + z},\ x + z,\ \frac{yz}{x + z} \right)$$

obtained from R_E by removing all monomials of degree 3 is an I-rational solution too.

Let PL_n denote the set of piecewise-linear functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Definition 5.2. By the *tropicalization* we mean the mapping $(\cdot)^t : I_n \to PL_n$ that is defined for functions $r = f/g \in I_n$, where

$$f = \sum_{i_1 + \ldots + i_n > 0} \alpha_{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n}, \quad g = \sum_{j_1 + \ldots + j_n \geq 0} \beta_{j_1 \ldots j_n} x_1^{j_1} \ldots x_n^{j_n},$$

by the rule

$$r^t = \begin{cases} \max\limits_{i_1 + \ldots + i_n > 0} \{ i_1 x_1 + \ldots + i_n x_n \} - \max\limits_{j_1 + \ldots + j_n > 0} \{ j_1 x_1 + \ldots + j_n x_n \} & \text{if } g \neq 1, \\ \max\limits_{i_1 + \ldots + i_n > 0} \{ i_1 x_1 + \ldots + i_n x_n \} & \text{if } g = 1. \end{cases}$$

The recursive procedure below allows us to obtain r^t from r.

Proposition 5.3. *Let $r = r(x_1, \ldots, x_n)$, $r_1 = r_1(x_1, \ldots, x_n)$, and $r_2 = r_2(x_1, \ldots, x_n)$ be rational functions in I_n.*

1. *If $r = x_i$ then $r^t = x_i$ for $i = 1, \ldots, n$.*
2. *We have $(r_1 + r_2)^t = \max\{r_1^t,\ r_2^t\}$.*
3. *We have $(r_1 r_2)^t = r_1^t + r_2^t$.*
4. *We have $\left(\dfrac{r_1}{r_2}\right)^t = r_1^t - r_2^t$.*

In particular, the definition of the tropicalization is well defined, i.e., if r_1 and r_2 are equal as rational functions then so are their tropicalizations.

Let $R(x_1, \dots, x_n) = (r_1, r_2, \dots, r_n) \in (I_n)^n$ be a rational vector-valued function in n variables. The *tropicalization* of R is defined component-wise as follows:

$$R^t(x_1, \dots, x_n) := (r_1^t(x_1, \dots, x_n), \dots, r_n^t(x_1, \dots, x_n)).$$

Example 5.4. The tropicalization of the electric solution R_E from Example 5.1 is the mapping

$$R_E^t(x, y, z) = (x + y - \max\{x, z, x + y + z\},\ \max\{x, z, x + y + z\},\ y + z - \max\{x, z, x + y + z\}).$$

The tropicalization of R_e from Example 5.1 is the mapping

$$R_e^t(x, y, z) = (x + y - \max\{x, z\},\ \max\{x, z\},\ y + z - \max\{x, z\}).$$

The mappings R_E^t and R_e^t are piecewise-linear solutions of the tetrahedron equation.

Remark 5.5. The solution R_E^t consists of three linear parts. We have

$$R_1(x, y, z) = (y, x, y + z - x),$$
$$R_2(x, y, z) = (x + y - z, z, y),$$
$$R_3(x, y, z) = (-z, x + y + z, -x).$$

We may regard these mappings as mappings on $\mathbb{R}^3$. It is not difficult to show that $(\mathbb{R}, R_1)$ and $(\mathbb{R}, R_2)$ are solutions of the tetrahedron equation; however, $(\mathbb{R}, R_3)$ is not a solution.

Definition 5.6. Let r_1 and r_2 be rational functions in $\mathbb{R}(x_1, \dots, x_n)$ and let $1 \leqslant k \leqslant n$. By the *k-composition* of r_1 and r_2 we mean the rational function $r_1 \circ_k r_2 \in \mathbb{R}(x_1, \dots, x_n)$ defined by the rule

$$(r_1 \circ_k r_2)(x_1, \dots, x_n) := r_1(x_1, \dots, x_{k-1}, r_2(x_1, \dots, x_n), x_{k+1}, \dots, x_n).$$

Proposition 5.7. *If $r_1, r_2 \in I_n$ then $(r_1 \circ_k r_2)^t = r_1^t \circ_k r_2^t$.*

Proof. We use the recursive procedure from Proposition 5.3. If $r_1(x_1, \dots, x_n) = x_i$, where x_i is a variable, then there is nothing to prove. If

$$r_1(x_1, \dots, x_n) = f(x_1, \dots, x_n) * g(x_1, \dots, x_n),$$

where f and g are rational functions in I_n and $* \in \{+, \cdot, /\}$, then

$$r_1^t(x_1, \dots, x_n) = f^t(x_1, \dots, x_n) *^t g^t(x_1, \dots, x_n),$$

where $*^t \in \{\max, +, -\}$, and

$$(r_1 \circ_k r_2)(x_1, \dots, x_n) = (f \circ_k r_2)(x_1, \dots, x_n) * (g \circ_k r_2)(x_1, \dots, x_n).$$

We find that

$$(r_1 \circ_k r_2)^t(x_1, \dots, x_n) = (f \circ_k r_2)^t(x_1, \dots, x_n) *^t (g \circ_k r_2)^t(x_1, \dots, x_n).$$

By induction, we obtain the required assertion. □

Corollary 5.8. *The tropicalization preserves the composition of rational vector-valued functions of such a form.*

Theorem 5.9. *If $(\mathbb{R}_{>0}, R)$ with $R \in (I_n)^n$ is a solution of the n-simplex equation on the set of positive real numbers then its tropicalization $(\mathbb{R}_{>0}, R^t)$ is a solution of the n-simplex equation too.*

Proof. For every $n \geqslant 2$, the n-simplex equation has the form

$$R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}} = R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}}.$$

We apply the tropicalization on both sides of this equality. We obtain

$$(R_{\overline{1}} R_{\overline{2}} \cdots R_{\overline{n+1}})^t = (R_{\overline{n+1}} \cdots R_{\overline{2}} R_{\overline{1}})^t.$$

By Corollary 5.8, we have

$$R_{\overline{1}}^t R_{\overline{2}}^t \cdots R_{\overline{n+1}}^t = R_{\overline{n+1}}^t \cdots R_{\overline{2}}^t R_{\overline{1}}^t.$$

This means that R^t is a solution of the n-simplex equation. □

5.2. *Formal tropicalization.* The following question naturally arises: Is it possible to obtain new solutions by using transforms that are similar to the tropicalization on a suitable class of solutions of the n-simplex equation? Surely, such transforms should exist. We call transforms that take solutions into new solutions *formal tropicalizations*. In the rigorous definition of a formal tropicalization, we need the notion of a partial algebra, i.e., a generalization of the notion of an universal algebra, where partial operations are allowed [18, Ch. 2]. A simple example of a partial algebra is a field, where the inverse with respect to multiplication is not defined for the zero element.

Definition 5.10. Let $\mathcal{A}$ be a partial algebra of signature Σ and let $R : A^n \to A^n$ be a partial function represented as a tuple $(f_1, \dots, f_n)$ of n-ary functions of a signature $\Sigma' \subset \Sigma$. Assume that (A, R) is a solution of the n-simplex equation. Let $\mathcal{B}$ be a partial algebra of signature Ξ and let g be a mapping that takes an n-ary function symbol of Σ' into an n-ary function of the signature Ξ. For every f_i, let $g(f_i)$ be the n-ary function of the signature Ξ, where each function symbol of Σ' is replaced with the corresponding n-ary function of the signature Ξ. The mapping $R^g : B^n \to B^n$ represented by $(g(f_1), \dots, g(f_n))$ is called a *formal tropicalization* of the solution R if (B, R^g) is a solution of the n-simplex equation.

The assertion "(A, R) is a solution of the n-simplex equation" can be regarded as N equations Φ_R in N variables such that, for every tuple of values of variables and every equation, either this equation holds or the value of the expression on one of its sides is undefined.

These equations may be regarded as formulas of the equational logic. For algebras, the equational logic was considered in [18, Appendix 4].

We adhere the above notation. Let T be a set of equational formulas of the signature Σ' such that $\mathcal{A}$ satisfies T and each formula in Φ_R can be deduced from T. We call T a set of *axioms* for the solution (A, R).

Proposition 5.11. *Let (A, R) be a solution of the n-simplex equation and let T be a set of its axioms. If g is a mapping such that $\mathcal{B}$ satisfies $g(T) := \{g(\phi) : \phi \in T\}$ then (B, R^g) is a formal tropicalization of (A, R), i.e., is a solution of the n-simplex equation.*

Proof. It is easy to see that the sets $g(\Phi_R)$ and Φ_{R^g} coincide. Since Φ_R can be deduced from T, we conclude that $g(\Phi_R)$ can be deduced from $g(T)$. Therefore, $\mathcal{B}$ satisfies Φ_{R^g}. □

Example 5.12. We consider the usual tropicalization of rational functions. We may regard rational solutions as solutions in the algebra $(X, \cdot^2, /^2, +^2, 1^0)$. Each solution can be deduced from the following set of axioms:

$$a \cdot b = b \cdot a, \quad a + b = b + a,$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad a + (b + c) = (a + b) + c, \quad a \cdot (b + c) = a \cdot b + a \cdot c,$$
$$a/b = a \cdot (1/b), \quad (1/a) \cdot (1/b) = 1/(a \cdot b), \quad a \cdot 1 = a, \quad a/1 = a, \quad a/a = 1.$$

Hence, there exists a mapping g to the signature of an arbitrary algebra that satisfies these axioms. One of such algebras is $(X, +^2, -^2, \max^2, 0^0)$. Since we regards multiplication by a fixed natural number as multiply applied addition operation, we may use the mapping g for rational solutions with natural coefficients and, in a sense, for rational solutions with positive rational coefficients.

Corollary 5.13. *Let $(\mathcal{A}, R)$ be a solution of the n-simplex equation and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ of the same class of signature Σ. Then $(\mathcal{B}, R)$ is a solution of the n-simplex equation.*

Proposition 5.14. *Let $(\mathcal{A}, R)$ be a solution of the n-simplex equation and let $\mathcal{B}$ be the homomorphic image of $\mathcal{A}$ with respect to a homomorphism h. Then $(\mathcal{B}, R^h)$ is a solution of the n-simplex equation.*

One of advantages of formal tropicalizations is possibility to regard solutions of the n-simplex equation as templates instead of fixed functions.

Example 5.15. As is known,

$$(x, y) \mapsto (ax, by + (1 - ab)x),$$

where $a, b \in \mathbb{Z}$, is a solution of the Yang–Baxter equation over $\mathbb{Z}$. Since $\mathbb{Z}$ is a generic $\mathbb{Z}$-module for the theory of left modules over commutative rings, the mapping

$$(x, y) \mapsto (ax, by + (1 - ab)x),$$

where R is a commutative ring and $a, b \in R$, is a solution of the Yang–Baxter equation over each left R-module.

The following assertion is immediate from Proposition 5.14.

Corollary 5.16. *Let $(\mathcal{A}, R)$ be a solution of the n-simplex equation and let T be a set of its axioms. We consider the subset T' of formulas in T with occurrences of variables and the class $\mathfrak{K}$ of algebras defined by T'. For an arbitrary algebra $\mathcal{B} \in \mathfrak{K}$, if there exists a homomorphism $h : \mathcal{A} \to \mathcal{B}$ then $(\mathcal{B}, R^h)$ is a solution of the n-simplex equation.*

Example 5.17. Let $\mathfrak{R}$ denote the class of commutative rings with the partial operation for taking the inverse element. Let $(\mathbb{R}, R)$ be a solution of the n-simplex equation of the form $(R_1, \dots, R_n)$, where

$$R_i = \frac{\sum a_j \cdot x_1^{\alpha_{j,1}} \cdot \ldots \cdot x_n^{\alpha_{j,n}}}{\sum b_k \cdot x_1^{\beta_{k,1}} \cdot \ldots \cdot x_n^{\beta_{k,n}}}, \quad \alpha_{j,m}, \beta_{k,m} \in \mathbb{N} \cup \{0\} \text{ for all } j, k, m.$$

In $\mathfrak{R}$, there is an obvious homomorphism $h : \mathbb{R} \to \mathcal{D}'[\mathbb{R}^m]$ from the field $\mathbb{R}$ of reals with the standard operations of addition and multiplication to the linear space $\mathcal{D}'[\mathbb{R}^m]$ of compactly supported distributions in $\mathbb{R}^m$ with the addition and convolution $*$ of functions. This homomorphism is defined by the rule

$$r \mapsto r \cdot \delta \in \mathcal{D}'[\mathbb{R}^m], \quad r \in \mathbb{R},$$

where δ is the Dirac delta-measure. Then $(\mathcal{D}'[\mathbb{R}^m], R^h)$ is a rational solution of the n-simplex equation. The components of the function R^h have the form

$$R_i^h = \left(\sum a_j \cdot x_1^{\alpha_{j,1}} * \ldots * x_n^{\alpha_{j,n}}\right) \left(\sum b_j \cdot x_1^{\beta_{j,1}} * \ldots * x_n^{\beta_{j,n}}\right)^{-1},$$

where the powers correspond to multiple application of the convolution and the inverse is considered with respect to the convolution.

6. SOLUTIONS OF THE PARAMETRIC YANG–BAXTER EQUATION

6.1. *Group extensions and the Yang–Baxter equation.* We consider a group extension

$$1 \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ j\ } K \longrightarrow 1$$

and a section $\varphi : K \to G$, i.e., a mapping with $j \circ \varphi = \mathrm{id}_K$. As is proven in [35], if K is an Abelian group then the conjugation quandle $\mathrm{Conj}(G)$ provides us with a solution of the Yang–Baxter equation

$$R_{12}^{a,b} R_{13}^{a,c} R_{23}^{b,c} = R_{23}^{b,c} R_{13}^{a,c} R_{12}^{a,b}, \quad a, b, c \in K,$$

on H with parameters in K.

We generalize this result. We represent elements of G as pairs $(x, a) \in H \times K$ and assume that $i(x) = (x, 1)$ and $j((x, a)) = a$. Then the multiplication on G regarded as a set of pairs is determined by the formula

$$(x, a) \circ (y, b) = (x \underset{a,b}{\circ} y, a \circ b) \text{ for a suitable } x \underset{a,b}{\circ} y \in H,$$

where $a \circ b$ is the product on K. We denote the multiplication on H by the same symbol $\circ$. This agrees with the formula

$$(x, 1) \circ (y, 1) = (x \circ y, 1).$$

We regard H as an algebra with the set of binary operations

$$\{\underset{a,b}{\circ} \mid a, b \in K\}.$$

By the axioms of groups, these operations satisfy the following axioms.

1. For all $x \in H$ and $a \in K$, there exists a unique $x^{-1}_{a,a^{-1}} = x^{-1}_{a^{-1},a} \in H$ such that

$$x^{-1}_{a^{-1},a} \underset{a^{-1},a}{\circ} x = x \underset{a,a^{-1}}{\circ} x^{-1}_{a,a^{-1}} = 1.$$

2. For all $x, y, z \in H$ and $a, b, c \in K$, we have

$$(x \underset{a,b}{\circ} y) \underset{a\circ b,c}{\circ} z = x \underset{a,b\circ c}{\circ} (y \underset{b,c}{\circ} z).$$

We generalize this construction to groups with a right distributive groupoid structure. We assume that a set G is endowed with a binary operation $* : G \times G \to G$ such that $(G, *)$ is a right distributive groupoid. Let a set H be closed with respect to $*$ and let this multiplication define a right distributive groupoid structure on K. As is known,

$$R(g, h) = (g, h * g), \quad g, h \in G,$$

is a solution of the Yang–Baxter equation on G. In [35], the operation $*$ was considered with $a * b = b^{-1}ab$ for $a, b \in G$, i.e., $(G, *)$ was the conjugation quandle.

We represent G as the set of pairs $(x, a) \in H \times K$. We have

$$(x, a) * (y, b) = (x \underset{a,b}{*} y, a * b) \text{ for a suitable } x \underset{a,b}{*} y \in H.$$

We obtain an operation $*$ on H and a set $\{\underset{a,b}{*} \mid a, b \in K\}$ of operations. The following assertion is obvious because $*$ is right distributive.

Lemma 6.1. *For all $x, y, z \in H$ and $a, b, c \in K$, we have*

$$(x \underset{a,b}{*} y) \underset{a*b,c}{*} z = (x \underset{a,c}{*} z) \underset{a*c,b*c}{*} (y \underset{b,c}{*} z).$$

We are ready to prove the following assertion.

Proposition 6.2. *For all $a, b, c \in K$, the equality*

$$R^{a,b}_{12}\, R^{a,c*b}_{13}\, R^{b,c}_{23} = R^{b*a,c*a}_{23}\, R^{a,c}_{13}\, R^{a,b}_{12}$$

holds in H, where

$$R^{u,v}(x, y) = (x, y \underset{v,u}{*} x), \quad u, v \in K.$$

Proof. Since (G, R) is a solution of the Yang–Baxter equation, for all $g, h, k \in G$, we have

$$R_{12}\, R_{13}\, R_{23}(g, h, k) = R_{23}\, R_{13}\, R_{12}(g, h, k).$$

Let $g = (x, a)$, $h = (y, b)$, and $k = (z, c)$. We consider the expression on the left-hand side:

$$\begin{aligned} &R_{12}\, R_{13}\, R_{23}((x, a), (y, b), (z, c)) = R_{12}\, R_{13}((x, a), (y, b), (z, c) * (y, b)) \\ &\quad = R_{12}\, R_{13}\left((x, a), (y, b), (z \underset{c,b}{*} y, c * b)\right) = R_{12}\left((x, a), (y, b), ((z \underset{c,b}{*} y) \underset{c*b,a}{*} x, (c * b) * a)\right) \\ &\quad = \left((x, a), (y \underset{b,a}{*} x, b * a), ((z \underset{c,b}{*} y) \underset{c*b,a}{*} x, (c * b) * a)\right). \end{aligned}$$

We consider the expression on the right-hand side:

$$\begin{aligned} &R_{23}\, R_{13}\, R_{12}((x, a), (y, b), (z, c)) = R_{23}\, R_{13}\left((x, a), (y \underset{b,a}{*} x, b * a), (z, c)\right) \\ &\quad = R_{23}\left((x, a), (y \underset{b,a}{*} x, b * a), (z \underset{c,a}{*} x, c * a)\right) \\ &\quad = \left((x, a), (y \underset{b,a}{*} x, b * a), ((z \underset{c,a}{*} x) \underset{c*a,b*a}{*} (y \underset{b,a}{*} x), (c * a) * (b * a))\right). \end{aligned}$$

We consider the restriction of R to $H \times H$ and put

$$R^{u,v}(x,y) = (x, y \underset{v,u}{*} x)$$

for $u, v \in K$. We obtain the required equality. □

Corollary 6.3. *If $(K, *)$ is a trivial right distributive groupoid (i.e., if $u * v = u$ for all $u, v \in K$) then, for all $a, b, c \in K$, the equality*

$$R_{12}^{a,b}\, R_{13}^{a,c}\, R_{23}^{b,c} = R_{23}^{b,c}\, R_{13}^{a,c}\, R_{12}^{a,b}$$

holds in H.

Example 6.4. We consider a group extension

$$1 \longrightarrow H \overset{i}{\longrightarrow} G \overset{j}{\longrightarrow} K \longrightarrow 1\,.$$

We assume that K is a group of exponent 2. As is well known, the group K is Abelian. For example, we may consider $K \simeq (\mathbb{Z}/2\mathbb{Z})^n$. On $(G, \circ)$, we consider the elementary solution $R : G^2 \to G^2$ of the Yang–Baxter equation defined by the rule

$$R(x,y) = (x, x \circ y^{-1} \circ x).$$

This solution corresponds to the core quandle $(\mathrm{Core}(G), *)$, where $y * x = x \circ y^{-1} \circ x$. We again regard G as the set of pairs (g, k), with $g \in H$ and $k \in K$. We represent the group operations on G as follows:

$$(g_1, k_1) \circ (g_2, k_2) = (g_1 \underset{k_1,k_2}{\circ} g_2, k_1 \circ k_2),$$
$$(g_1, k_1)^{-1} = (g^{-1}_{k,k^{-1}}, k_1^{-1}),$$

where $\underset{a,b}{\circ} : H^2 \to H$ denotes an operation on H that depends on parameters a and b and $g^{-1}_{k,k^{-1}}$ denotes the inverse element with respect to $\underset{k,k^{-1}}{\circ}$. We define a mapping $R^{a,b} : H^2 \to H^2$. We put

$$R^{a,b}(x,y) = \left(x, (x \underset{a,b^{-1}}{\circ} y^{-1}_{b^{-1},b}) \underset{ab^{-1},a}{\circ} x \right).$$

Then R is a solution of the parametric Yang–Baxter equation

$$R_{12}^{a,b} R_{13}^{a,c} R_{23}^{b,c} = R_{23}^{b,c} R_{13}^{a,c} R_{12}^{a,b}.$$

Remark 6.5. As is known, on each group G, we can introduce two distinct structures of a verbal quandle; namely, $\mathrm{Core}(G)$, where $g * h = hg^{-1}h$, and $\mathrm{Conj}_n(G)$, where $g *_n h = h^{-n} g h^n$ for a natural n. The case of $\mathrm{Core}(G)$ was considered in the previous example. The case of $\mathrm{Conj}_1(G)$ was studied in [35]. The description of the case of $\mathrm{Conj}_n(G)$ is similar. The quandle $\mathrm{Conj}_n(G)$ provides us with a solution of the parametric Yang–Baxter equation if the subgroup K^n of K generated by nth powers of elements forms a subset of the center $Z(K)$ of K.

6.2. *Representations of virtual braid groups and the Yang–Baxter equation.* In [2, 5], representations were constructed of the virtual braid group VB_n. In [5], a representation $\varphi : VB_n \longrightarrow Aut(F_{n,3n})$ was constructed to the automorphism group of the free product $F_{n,3n} = F_n * \mathbb{Z}^{3n}$, where $F_n = \langle x_1 \ldots x_n \rangle$ is a free group of rank n and $\mathbb{Z}^{3n} = \langle w_1, \ldots, w_n, u_1, \ldots, u_n, v_1, \ldots, v_n \rangle$ is a free Abelian group of rank $3n$. It is determined by the following action on the generators:

$$\varphi(\sigma_i) : \begin{cases} x_i \longrightarrow x_i x_{i+1}^{u_i} x_i^{-w_{i+1} u_{i+1}}, \\ x_{i+1} \longrightarrow x_i^{w_{i+1}}, \end{cases} \qquad \varphi(\sigma_i) : \begin{cases} w_i \longrightarrow w_{i+1}, \\ w_{i+1} \longrightarrow w_i, \end{cases}$$

$$\varphi(\sigma_i) : \begin{cases} u_i \longrightarrow u_{i+1}, \\ u_{i+1} \longrightarrow u_i, \end{cases} \qquad \varphi(\sigma_i) : \begin{cases} v_i \longrightarrow v_{i+1}, \\ v_{i+1} \longrightarrow v_i, \end{cases}$$

$$\varphi(\rho_i):\begin{cases} x_i \longrightarrow x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \longrightarrow x_i^{v_{i+1}}, \end{cases} \qquad \varphi(\rho_i):\begin{cases} w_i \longrightarrow w_{i+1}, \\ w_{i+1} \longrightarrow w_i, \end{cases}$$

$$\varphi(\rho_i):\begin{cases} u_i \longrightarrow u_{i+1}, \\ u_{i+1} \longrightarrow u_i, \end{cases} \qquad \varphi(\rho_i):\begin{cases} v_i \longrightarrow v_{i+1}, \\ v_{i+1} \longrightarrow v_i. \end{cases}$$

Theorem 6.6. *Let $G = B * A$, where A and B are groups and A is an Abelian group. Then the mappings*

$$R_{12}^{u,v,w}(x,y,z) = (x^w, xy^u x^{-wv}, z),$$
$$R_{13}^{u,p,q}(x,y,z) = (x^u, y, xz^u x^{-pq}),$$
$$R_{23}^{v,p,q}(x,y,z) = (x, y^q, yz^v y^{-pq})$$

give rise to a solution of the parametric Yang–Baxter equation

$$R_{12}^{u,v,w} R_{13}^{u,p,q} R_{23}^{v,p,q} = R_{23}^{v,p,q} R_{13}^{u,p,q} R_{12}^{u,v,w}$$

on B with parameters $u, v, w, p, q, u_1, v_1, u_2, v_2 \in A$.

The mappings

$$T_{12}^{u,v}(x,y,z) = (x^u, y^v, z),$$
$$T_{13}^{u_1,v_1}(x,y,z) = (x^{u_1}, y, z^{v_1}),$$
$$T_{23}^{u_2,v_2}(x,y,z) = (x, y^{u_2}, z^{v_2})$$

give rise to a solution of the parametric Yang–Baxter equation

$$T_{12}^{u,v} T_{13}^{u_1,v_1} T_{23}^{u_2,v_2} = T_{23}^{u_2,v_2} T_{13}^{u_1,v_1} T_{12}^{u,v}$$

on B with parameters in A.

Proof. Let $G = B * A$. We define $R, T : G \times G \to G \times G$. We put

$$R^{u,v,w}(x,y) = (x^w, xy^u x^{-wv}), \quad T^{u,v}(x,y) = (x^u, y^v)$$

for $x, y \in B$ and $u, v, w \in A$.

We construct a mapping R that is a solution of the parametric Yang–Baxter equation

$$R_{12}^{u,v,w} R_{13}^{u_1,v_1,w_1} R_{23}^{u_2,v_2,w_2} = R_{23}^{u_2,v_2,w_2} R_{13}^{u_1,v_1,w_1} R_{12}^{u,v,w},$$

where

$$R_{12}^{u,v,w}(x,y,z) = (x^w, xy^u x^{-wv}, z),$$
$$R_{13}^{u_1,v_1,w_1}(x,y,z) = (x^{w_1}, y, xz^{u_1} x^{-w_1 v_1}),$$
$$R_{23}^{u_2,v_2,w_2}(x,y,z) = (x, y^{w_2}, yz^{u_2} y^{-w_2 v_2}).$$

We consider the action of the expression on the right-hand side. We obtain

$$\begin{aligned} R_{12}^{u,v,w} R_{13}^{u_1,v_1,w_1} R_{23}^{u_2,v_2,w_2}(x,y,z) &= R_{12}^{u,v,w} R_{13}^{u_1,v_1,w_1}(x, y^{w_2}, yz^{u_2} y^{-w_2 v_2}) \\ &= R_{12}^{u,v,w}(x^{w_1}, y^{w_2}, x(yz^{u_2} y^{-w_2 v_2})^{u_1} x^{-w_1 v_1}) \\ &= (x^{w_1 w}, x^{w_1} y^{w_2 u} x^{-w_1 w v}, xy^{u_1} z^{u_2 u_1} y^{-w_2 v_2 u_1} x^{-w_1 v_1}). \end{aligned}$$

We consider the action of the expression on the left-hand side. We obtain

$$\begin{aligned} R_{23}^{u_2,v_2,w_2} R_{13}^{u_1,v_1,w_1} R_{12}^{u,v,w}(x,y,z) &= R_{23}^{u_2,v_2,w_2} R_{13}^{u_1,v_1,w_1}(x^w, xy^u x^{-wv}, z) \\ &= R_{23}^{u_2,v_2,w_2}(x^{ww_1}, xy^u x^{-wv}, x^w z^{u_1} x^{-ww_1 v_1}) \\ &= (x^{ww_1}, x^{w_2} y^{uw_2} x^{-wvw_2}, xy^u x^{-wv+wu_2} z^{u_1 u_2} x^{-ww_1 v_1 u_2 + wvw_2 v_2} y^{-uw_2 v_2} x^{-w_2 v_2}). \end{aligned}$$

Since x, y, and z are arbitrary elements of B, we obtain the following system of equations:

$$\begin{cases} ww_1 = w_1w, \;\; w_1 = w_2, \;\; uw_2 = w_2u, \;\; wvw_2 = w_1wv, \\ u = u_1, \;\; w(u_2 - v) = 0, \;\; u_1u_2 = u_2u_1, \;\; w_2v_2 = w_1v_1, \\ w(vw_2v_2 - w_1v_1u_2) = 0, \;\; uv_2w_2 = w_2v_2u_1. \end{cases}$$

This system implies that $w_1 = w_2$, $u = u_1$, $u_2 = v$, and $v_2 = v_1$. Hence, there are three free parameters for $R_{12}^{u,v,w}$ and two free parameters for $R_{13}^{u_1,v_1,w_1} = R_{13}^{u,v_1,w_1}$, while $R_{23}^{u_2,v_2,w_2} = R_{23}^{v,v_1,w_1}$ lacks free parameters.

Thus, we obtain a solution of the parametric Yang–Baxter equation

$$R_{12}^{u,v,w} R_{13}^{u,p,q} R_{23}^{v,p,q} = R_{23}^{v,p,q} R_{13}^{u,p,q} R_{12}^{u,v,w}.$$

We consider the mapping $T : G \times G \to G \times G$ defined by the rule $T^{u,v}(x, y) = (x^u, y^v)$. Then

$$T_{12}^{u,v}(x, y, z) = (x^u, y^v, z),$$
$$T_{13}^{u_1,v_1}(x, y, z) = (x^{u_1}, y, z^{v_1}),$$
$$T_{23}^{u_2,v_2}(x, y, z) = (x, y^{u_2}, z^{v_2})$$

are mappings on G^3. We prove that they satisfy the parametric Yang–Baxter equation.

We consider the action of the expression on the right-hand side. We obtain

$$T_{12}^{u,v} T_{13}^{u_1,v_1} T_{23}^{u_2,v_2}(x, y, z) = T_{12}^{u,v} T_{13}^{u_1,v_1}(x, y^{u_2}, z^{v_2}) = T_{12}^{u,v}(x^{u_1}, y^{u_2}, z^{v_2v_1}) = (x^{u_1u}, y^{u_2v}, z^{v_2v_1}).$$

We consider the action of the expression on the left-hand side. We obtain

$$T_{23}^{u_2,v_2} T_{13}^{u_1,v_1} T_{12}^{u,v}(x, y, z) = T_{23}^{u_2,v_2} T_{13}^{u_1,v_1}(x^u, y^v, z) = T_{23}^{u_2,v_2}(x^{uu_1}, y^v, z^{v_1}) = (x^{uu_1}, y^{vu_2}, z^{v_1v_2}).$$

Therefore, we have six free parameters. □

7. SOLUTIONS OF THE PARAMETRIC n-SIMPLEX EQUATION

We generalize the results of the previous section. As above, we assume that G is an extension of a subgroup H by a group K and (G, R) and (K, r) are solutions of the n-simplex equation on G and K respectively such that the diagram

$$\begin{array}{ccc} G^n & \xrightarrow{j^n} & K^n \\ \big\downarrow{\scriptstyle R} & & \big\downarrow{\scriptstyle r} \\ G^n & \xrightarrow{j^n} & K^n \end{array}$$

commutes, where G^n and K^n are the Cartesian products.

Each element $x \in G$ is represented by a pair (h, k) such that $k = j(x) \in K$, $h = i^{-1}(x \cdot (\varphi(k))^{-1}) \in H$, where i is an embedding of H into G, and $i(h) = (h, 1)$. We represent

$$R(x_1, \dots, x_n) = R((h_1, k_1), (h_2, k_2), \dots (h_n, k_n)) = (y_1, y_2, \dots, y_n) \in G^n.$$

We represent the expression on the right-hand side as follows:

$$(y_1, y_2, \dots, y_n) = ((h'_1, k'_1), (h'_2, k'_2), \dots (h'_n, k'_n)) \text{ for some } h'_i \in H \text{ and } k'_i \in K.$$

We put

$$R^{k_1,\dots,k_n}(h_1, \dots, h_n) = (h'_1, h'_2, \dots h'_n),$$
$$r(k_1, \dots, k_n) = (k'_1, k'_2, \dots k'_n).$$

We rewrite R as follows:

$$R(x_1, \dots, x_n) = (R^{k_1,\dots,k_n}(h_1, \dots, h_n), r(k_1, \dots, k_n)).$$

Since R is a solution of the n-simplex equation on G, the family $R^{k_1,\dots,k_n}$ of functions is a solution of the parametric n-simplex equation on H.

Proposition 7.1. *Let (G, R) be a solution of the n-simplex equation. We define*

$$s_m(k_1, \ldots, k_N) = p^{\overline{m}}(r_{\overline{m-1}} \circ \ldots \circ r_{\overline{1}}(k_1, \ldots, k_N)),$$
$$z_m(k_1, \ldots, k_N) = p^{\overline{m}}(r_{\overline{m+1}} \circ \ldots \circ r_{\overline{n+1}}(k_1, \ldots, k_N)),$$

where $p^{\overline{m}}$ denotes the projection from K^N to K^n whose kernel consists of components with numbers outside the multi-index $\overline{m}$. Then $(H, R^{k_1, \cdots, k_n})$ is a solution of the parametric n-simplex equation

$$R_{\overline{1}}^{z_1(k)} R_{\overline{2}}^{z_2(k)} \cdots R_{\overline{n+1}}^{z_{n+1}(k)} = R_{\overline{n+1}}^{s_{n+1}(k)} \cdots R_{\overline{2}}^{s_2(k)} R_{\overline{1}}^{s_1(k)},$$

where $k = (k_1, \ldots, k_N) \in K^N$ is a vector of parameters.

If $r = \mathrm{id}$, i.e., the identity mapping, then we obtain

$$R_{\overline{1}}^{k_{\overline{1}}} R_{\overline{2}}^{k_{\overline{2}}} \cdots R_{\overline{n+1}}^{k_{\overline{n+1}}} = R_{\overline{n+1}}^{k_{\overline{n+1}}} \cdots R_{\overline{2}}^{k_{\overline{2}}} R_{\overline{1}}^{k_{\overline{1}}}.$$

Notice that if $r(k_0, \ldots, k_0) = (k_0, \ldots, k_0)$ for some $k_0 \in K$ then $(H, R^{k_0, \cdots, k_0})$ is a solution of the n-simplex equation.

7.1. *Solutions of the n-simplex equation on group extensions.* We consider a group extension

$$1 \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ j\ } K \longrightarrow 1\,.$$

We assume that (H, R) and (K, T) are solutions of the n-simplex equation on H and K respectively.

Definition 7.2. We say that (G, Q) is an *extension of the solution (H, R) by the solution (K, T)* if (G, Q) is a solution of the n-simplex equation and the diagram

$$\begin{array}{ccccc} H^n & \xrightarrow{\ i^n\ } & G^n & \xrightarrow{\ j^n\ } & K^n \\ \big\downarrow{\scriptstyle R} & & \big\downarrow{\scriptstyle Q} & & \big\downarrow{\scriptstyle T} \\ H^n & \xrightarrow{\ i^n\ } & G^n & \xrightarrow{\ j^n\ } & K^n \end{array}$$

commutes. We identify the group H^n with its image under i^n in G^n.

Proposition 7.3. *If (H, R) and (K, T) are solutions of the n-simplex equation then there exists a solution (G, Q) of the n-simplex equation that is an extension of (H, R) by (K, T).*

Proof. We consider the extension of the Cartesian powers

$$1 \longrightarrow H^n \xrightarrow{\ i^n\ } G^n \underset{\varphi}{\overset{j^n}{\rightleftarrows}} K^n \longrightarrow 1.$$

For every $g \in G^n$, there exists a unique representation $g = h_g k_g$, where

$$k_g = \varphi \circ j^n(k), \qquad h_g = g k_g^{-1},$$

and $h_g \in H^n$. Therefore, we represent the set G^n as the Cartesian product of the sets H^n and K^n. We define

$$Q(g) = R(h_g)\varphi\left(T j^n(g)\right).$$

In terms of the Cartesian product above, we represent this function as $(R(h_g), \varphi T j^n(k_g))$. Notice that a function F defined on $A \times B$ for some sets A and B is a solution if and only if its projections are solutions. We conclude that Q is an extension of solutions if and only if $\varphi T j^n(g)$ is a solution on G.

We put $\varphi = \tilde{\varphi}^n$, where $\tilde{\varphi}$ is a section of $G \xrightarrow{j} K$. We obtain a solution because the equality

$$(\tilde{\varphi}^n T j^n)_{\overline{1}} \ldots (\tilde{\varphi}^n T j^n)_{\overline{n+1}} = (\tilde{\varphi}^n T j^n)_{\overline{n+1}} \ldots (\tilde{\varphi}^n T j^n)_{\overline{1}}$$

is equivalent to the equality

$$\tilde{\varphi}^N T_{\bar{1}}\, j^N \dots \tilde{\varphi}^N T_{\overline{n+1}}\, j^N = \tilde{\varphi}^N T_{\overline{n+1}}\, j^N \dots \tilde{\varphi}^N T_{\bar{1}}\, j^N.$$

Since $j \circ \tilde{\varphi} = \mathrm{id}$, the above equality holds if and only if (K, T) is a solution. □

We illustrate the use of Proposition 7.3 for constructing new solutions.

Example 7.4. We consider the additive group $(\mathbb{Z}, +)$ of integers as an extension of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \mathbb{Z} \xrightarrow{\ j\ } \mathbb{Z}_p \longrightarrow 0.$$

Here i denotes the multiplication by p and, for every $b \in \mathbb{Z}$, we denote by $j(b)$ the remainder in the division of b by p. For simplicity, we denote $\bar{b} := j(b)$ and if $a = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ then we write $\bar{a} := (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) \in \mathbb{Z}_p^n$.

Let R and T be solutions of the n-simplex equation on $\mathbb{Z}$ and $\mathbb{Z}_p$ respectively. We fix a set-theoretical section $\varphi : \mathbb{Z}_p \to \mathbb{Z}$. For each $b \in \mathbb{Z}$, there is a unique representation

$$(b - \varphi(\bar{b}),\ \bar{b}) \in \mathbb{Z} \times \mathbb{Z}_p.$$

Hence, the mapping

$$a \mapsto R(a - \varphi^n(\bar{a})) + \varphi^n T(\bar{a})$$

is a solution of the n-simplex equation on $\mathbb{Z}$.

For example, let $n = 3$. Assume that $R = \mathrm{id}$ and $T : (x, y, z) \mapsto (x + 2y - 2z, 2z - y, z)$ are solutions of the tetrahedron equation on $\mathbb{Z}$ and $\mathbb{Z}_p$ respectively. Then the mapping

$$(x, y, z) \mapsto \left(\overline{(x + 2y - 2z)} + x - \bar{x}, \overline{(2z - y)} + y - \bar{y}, z\right)$$

is a new solution of the tetrahedron equation on $\mathbb{Z}$, where the section embeds $\mathbb{Z}_p$ into $\mathbb{Z}$ as the first p nonnegative elements.

8. INVERSE LIMIT OF SOLUTIONS

Let $\mathcal{A}$ denote a category of algebras such that the n-simplex equation makes sense. For example, we may consider the category of groups, rings, modules, etc. We define a category $\mathcal{CA}(n)$. The objects are pairs (X, R_X), where X is an object of $\mathcal{A}$ and R_X is a solution of the n-simplex equation. Morphisms of $\mathcal{CA}(n)$ are morphisms of $\mathcal{A}$ such that

$$\mathrm{Mor}((X, R_X), (Y, R_Y)) = \{f \in \mathrm{Mor}(X, Y) \mid f \circ R_X = R_Y \circ f\}.$$

We describe solutions on the inverse limit. For the definition of the inverse limit and its properties, the reader is referred to [33, Ch. 3]. Let D be a small category and let $\mathcal{F} : D \to \mathcal{CA}(n)$ be a diagram. We apply the forgetful functor $T : \mathcal{CA}(n) \to \mathcal{A}$ and obtain a diagram in $\mathcal{A}$.

Proposition 8.1. *If the inverse limit* $\lim(T \circ \mathcal{F})$ *exists in* $\mathcal{A}$ *then the inverse limit* $\lim \mathcal{F}$ *exists in* $\mathcal{CA}(n)$.

Proof. Since the limits commute, we have

$$\lim_{a \in D} \mathcal{F}(a)^n = (\lim_{a \in D} \mathcal{F}(a))^n.$$

Let $X = \lim(T \circ \mathcal{F})^n$ and let F denote $\mathcal{F}^n$. For an arbitrary object $A \in D$, let $p_A \in \mathrm{Mor}(X, F(A))$ be the morphism from the definition of the limit. We define a mapping R on X. We put

$$(x_A)_{A \in D} \mapsto (R_A(x_A))_{A \in D}.$$

We need to prove that $R(X) \subset X$. Assume that there exists $x \in X$ with $R(x) \notin X$. Then there exists a morphism f_{AB} such that

$$f_{AB} \circ p_A(R(x)) \neq p_B(R(x))$$

or, which is equivalent,

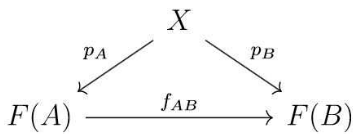
$$f_{AB} \circ R_A(x_A) \neq R_B(x_B).$$

Conditions on morphisms imply

$$R_B(f_{AB}(x_A)) \neq R_B(x_B).$$

Since $f(x_A) = x_B$, we arrive at a contradiction.

For objects $A, B \in D$ and a morphism $f \in \mathrm{Mor}(A, B)$, the diagram

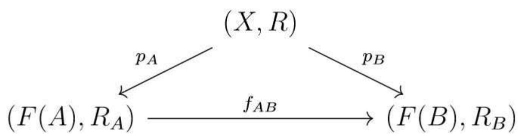


commutes, where $f_{AB} = F(f)$.

Since $x_A = p_A(x)$, the condition on morphisms yields

$$f_{AB} \circ R_A \circ p_A(x) = R_B \circ f_{AB} \circ p_A(x) = R_B \circ p_B(x).$$

We conclude that the diagram



commutes too. □

Corollary 8.2. *We regard p-adic integers $\mathbf{Z}_p$ as the inverse limit of $\mathbb{Z}_{p^k}$ and assume that $(\mathbb{Z}_{p^k}, R_k)$ with $k = 1, 2, \ldots$ are solutions of the n-simplex equation such that $R_k \circ (\phi_k)^{\times n} = R_{k-1}$, where ϕ_k is the projection from $\mathbb{Z}_{p^k}$ to $\mathbb{Z}_{p^{k-1}}$. Then this family determines a solution $(\mathbf{Z}_p, R)$ of the n-simplex equation.*

Example 8.3. We consider the family of linear solutions

$$R_k(x, y, z) = (a_k x, (1 - a_k b_k)x + b_k y, \quad (1 - a_k b_k)c_k x + (1 - b_k c_k)y + c_k z)$$

of the tetrahedron equation on $\mathbb{Z}_{p^k}$. We also assume that the coefficients satisfy the conditions

$$a_{k+1} = a_k \pmod p)^k, \quad b_{k+1} = b_k \pmod p)^k, \quad c_{k+1} = c_k \pmod p)^k.$$

The family R_k generates a solution R on $\mathbf{Z}_p$, where

$$R(x, y, z) = (ax, (1 - ab)x + by, \quad (1 - ab)cx + (1 - bc)y + cz),$$

and a, b, and c are p-adic numbers defined by the sequences

$$a = (a_1, \ldots, a_k, \ldots), \quad b = (b_1, \ldots, b_k, \ldots), \quad c = (c_1, \ldots, c_k, \ldots).$$

9. TETRAHEDRON EQUATION AND CORRESPONDING ALGEBRAS

In the present section, we consider the tetrahedron equation

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}.$$

By a solution we mean a set-theoretical solution of this equation.

9.1. 3-*Ternoinds*. An algebra with a single ternary operation is called a *ternar*. An algebra with k ternary operations is called a k-*ternoid*. If $R = (f, g, h) : X^3 \to X^3$ is a solution of the tetrahedron equation on a set X then we define three ternary operations $[\cdot,\cdot,\cdot]_i : X^3 \to X$ with $i = 1, 2, 3$ on X. We put

$$[a,b,c]_1 = f(a,b,c), \quad [a,b,c]_2 = g(a,b,c), \quad [a,b,c]_3 = h(a,b,c),$$

where $a, b, c \in X$. For simplicity, we distinguish between operations by using distinct types of brackets and write

$$[a,b,c]_1 = [a,b,c], \quad [a,b,c]_2 = \langle a,b,c\rangle, \quad [a,b,c]_3 = \{a,b,c\}.$$

Straightforward calculations prove the following assertion.

Proposition 9.1. *Let $(X, [\cdot,\cdot,\cdot], \langle\cdot,\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a 3-ternoid. It determines a solution of the tetrahedron equation if and only if the following six equations hold for all $(x, y, z, t, p, q) \in X^6$:*

$$[[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]] = [[x, y, z], t, p],$$
$$\langle [x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\rangle = [\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q],$$
$$\{[x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle], [y, t, \{z, p, q\}], [z, p, q]\}$$
$$= [\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}],$$
$$\langle x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\rangle = \langle\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\rangle,$$
$$\{x, \langle y, t, \{z, p, q\}\rangle, \langle z, p, q\rangle\} = \langle\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}\rangle,$$
$$\{y, t, \{z, p, q\}\} = \{\{x, y, z\}, \{[x, y, z], t, p\}, \{\langle x, y, z\rangle, \langle [x, y, z], t, p\rangle, q\}\}.$$

The following three corollaries describe elementary solutions of the tetrahedron equation.

Corollary 9.2. *Let $(X, [\cdot,\cdot,\cdot])$ be a ternar. Then the mapping $R : X^3 \to X^3$ defined by the rule*

$$R(a,b,c) = ([a,b,c], b, c), \quad a, b, c \in X,$$

satisfies the tetrahedron equation if and only if

$$[[x, t, p], [y, t, q], [z, p, q]] = [[x, y, z], t, p]$$

for all $x, y, z, t, p, q \in X$.

Corollary 9.3. *Let $(X, [\cdot,\cdot,\cdot])$ be a ternar. Then the mapping $R : X^3 \to X^3$ defined by the rule*

$$R(a,b,c) = (a, [a,b,c], c), \quad a, b, c \in X,$$

satisfies the tetrahedron equation if and only if

$$[[x, y, z], [x, t, p], q] = [x, [y, t, q], [z, p, q]] \tag{9.1}$$

for all $x, y, z, t, p, q \in X$.

Corollary 9.4. *Let $(X, [\cdot,\cdot,\cdot])$ be a ternar. Then the mapping $R : X^3 \to X^3$ defined by the rule*

$$R(a,b,c) = (a, b, [a,b,c]), \quad a, b, c \in X,$$

satisfies the tetrahedron equation if and only if

$$[[x, y, z], [x, t, p], [y, t, q]] = [y, t, [z, p, q]] \tag{9.2}$$

for all $x, y, z, t, p, q \in X$.

Remark 9.5. We notice that the equation in Corollary 9.3 is "more symmetric" than equations in Corollaries 9.2 and 9.4. In particular, the number of variables is the same on the left-hand and right-hand sides, while the number of variables on distinct sides of the remaining equations is distinct.

We call solutions from Corollary 9.2 *elementary solutions of the first kind* or 1-*elementary solutions*; we call solutions from Corollary 9.3 *elementary solutions of the second kind* or 2-*elementary solutions*; we call solutions from Corollary 9.4 *elementary solutions of the third kind* or 3-*elementary solutions*.

Let $P_{13} : X^3 \to X^3$ denote the permutation of the first and third components, i.e., put $P_{13}(x, y, z) = (z, y, x)$. If R is an 1-elementary solution then $P_{13}RP_{13}$ is a 3-elementary solution and vice versa. Therefore, we only need to consider 1- and 2-elementary solutions.

9.2. 1-*Elementary solutions.* We study 1-elementary solutions of the tetrahedron equation. We introduce an algebra $(X, \bar{*}, \bar{\circ}, \bar{\lhd}, \bar{\rhd})$ with four binary operations satisfying the relations

$$\begin{gathered} x \,\bar{\circ}\, y = (x \,\bar{\lhd}\, z) \,\bar{\circ}\, (y \,\bar{\lhd}\, z), \\ (x \,\bar{\circ}\, y) \,\bar{*}\, (z \,\bar{\circ}\, w) = (x \,\bar{*}\, z) \,\bar{\circ}\, (y \,\bar{*}\, w), \\ (x \,\bar{\rhd}\, y) \,\bar{\rhd}\, z = (x \,\bar{\rhd}\, z) \,\bar{\rhd}\, (y \,\bar{*}\, z), \\ (x \,\bar{*}\, y) \,\bar{\lhd}\, z = x \,\bar{\rhd}\, (y \,\bar{\circ}\, z). \end{gathered}$$

We call this algebra the *first tetrahedral* 4-*groupoid* or a T_1-*groupoid*.

Proposition 9.6. *A T_1-groupoid generates a 1-elementary solution (X, R) of the tetrahedron equation according to the rule*

$$R(x, y, z) = (x \,\bar{\rhd}\, (y \,\bar{\circ}\, z),\ y,\ z), \quad x, y, z \in X.$$

Proof. We prove that R is a solution. It suffices to show that R satisfies the equality in Corollary 9.2. This equality assumes the form

$$(x \,\bar{\rhd}\, (t \,\bar{\circ}\, p)) \,\bar{\rhd}\, ((y \,\bar{\rhd}\, (t \,\bar{\circ}\, q)) \,\bar{\circ}\, (z \,\bar{\rhd}\, (p \,\bar{\circ}\, q))) = (x \,\bar{\rhd}\, (y \,\bar{\circ}\, z)) \,\bar{\rhd}\, (t \,\bar{\circ}\, p).$$

We consecutively apply the relations to the expression on the left-hand side. We obtain

$$\begin{aligned} (x \,\bar{\rhd}\, (t \,\bar{\circ}\, p)) \,\bar{\rhd}\, ((y \,\bar{\rhd}\, (t \,\bar{\circ}\, q)) \,\bar{\circ}\, (z \,\bar{\rhd}\, (p \,\bar{\circ}\, q))) &= (x \,\bar{\rhd}\, (t \,\bar{\circ}\, p)) \,\bar{\rhd}\, (((y \,\bar{*}\, t) \,\bar{\lhd}\, q) \,\bar{\circ}\, ((z \,\bar{*}\, p) \,\bar{\lhd}\, q)) \\ &= (x \,\bar{\rhd}\, (t \,\bar{\circ}\, p)) \,\bar{\rhd}\, ((y \,\bar{*}\, t) \,\bar{\circ}\, (z \,\bar{*}\, p)) = (x \,\bar{\rhd}\, (t \,\bar{\circ}\, p)) \,\bar{\rhd}\, ((y \,\bar{\circ}\, z) \,\bar{*}\, (t \,\bar{\circ}\, p)) \\ &= (x \,\bar{\rhd}\, (y \,\bar{\circ}\, z)) \,\bar{\rhd}\, (t \,\bar{\circ}\, p), \end{aligned}$$

i.e., the required equality holds. □

Example 9.7. We consider an algebra $(V, \bar{*}, \bar{\circ}, \bar{\lhd}, \bar{\rhd})$ on a vector space V, where

$$\begin{aligned} x \,\bar{*}\, y &:= \beta x + (1 - \beta) y, \\ x \,\bar{\circ}\, y &:= x - y, \\ x \,\bar{\lhd}\, y &:= x + (\beta - 1) y, \\ x \,\bar{\rhd}\, y &:= \beta x + (1 - \beta) y, \end{aligned}$$

and β is an endomorphism of V. This system generates the solution

$$R(x, y, z) = (\beta x + (1 - \beta) y + (\beta - 1) z, y, z), \quad x, y, z \in V.$$

There exists 2-groupoids that give rise to 1-elementary solutions in a similar way. We consider an algebra $(X, \bar{*}, \bar{\circ})$ such that

$$\begin{gathered} x \,\bar{\circ}\, y = (x \,\bar{\circ}\, z) \,\bar{\circ}\, (y \,\bar{\circ}\, z), \\ (x \,\bar{*}\, y) \,\bar{*}\, z = (x \,\bar{*}\, z) \,\bar{*}\, (y \,\bar{*}\, z), \\ (x \,\bar{\circ}\, y) \,\bar{*}\, (z \,\bar{\circ}\, w) = (x \,\bar{*}\, z) \,\bar{\circ}\, (y \,\bar{*}\, w). \end{gathered}$$

We call this algebra the *first reduced tetrahedral* 4-*groupoid* or a *reduced* T_1-*groupoid*.

Proposition 9.8. *A reduced T_1-groupoid generates a 1-elementary solution (X, R) of the tetrahedron equation, where*

$$R(x, y, z) = (x \,\bar{*}\, (y \,\bar{\circ}\, z),\ y,\ z), \quad x, y, z \in X.$$

Proof. Notice that R is a solution if it satisfies the equation from Corollary 9.2 which assumes the form

$$(x \,\bar{*}\, (t \,\bar{\circ}\, p)) \,\bar{*}\, ((y \,\bar{*}\, (t \,\bar{\circ}\, q)) \,\bar{\circ}\, (z \,\bar{*}\, (p \,\bar{\circ}\, q))) = (x \,\bar{*}\, (y \,\bar{\circ}\, z)) \,\bar{*}\, (t \,\bar{\circ}\, p).$$

We consecutively apply the relations to the expression on the left-hand side and obtain

$$\begin{aligned} (x \,\bar{*}\, (t \,\bar{\circ}\, p)) \,\bar{*}\, ((y \,\bar{*}\, (t \,\bar{\circ}\, q)) \,\bar{\circ}\, (z \,\bar{*}\, (p \,\bar{\circ}\, q))) &= (x \,\bar{*}\, (t \,\bar{\circ}\, p)) \,\bar{*}\, ((y \,\bar{\circ}\, z) \,\bar{*}\, ((t \,\bar{\circ}\, q) \,\bar{\circ}\, (p \,\bar{\circ}\, q))) \\ &= (x \,\bar{*}\, (t \,\bar{\circ}\, p)) \,\bar{*}\, ((y \,\bar{\circ}\, z) \,\bar{*}\, (t \,\bar{\circ}\, p)) = (x \,\bar{*}\, (y \,\bar{\circ}\, z)) \,\bar{*}\, (t \,\bar{\circ}\, p), \end{aligned}$$

i.e., the required equality holds. □

Example 9.9. We consider an algebra $(V, \overline{*}, \overline{\circ})$ on a vector space V, where

$$x \mathbin{\overline{*}} y := \beta x + (1-\beta)y,$$
$$x \mathbin{\overline{\circ}} y := x - y,$$

and β is a fixed endomorphism of V. This system generates the 1-elementary solution

$$R(x,y,z) = (\beta x + (1-\beta)y + (\beta - 1)z, y, z), \quad x,y,z \in V.$$

Notice that this algebra is similar to the algebra from Example 9.7. In general, if $x \mathbin{\overline{*}} y = x \mathbin{\overline{\triangleright}} y$ in a T_1-groupoid then we remove the operations $x \mathbin{\overline{\triangleleft}} y$ and $x \mathbin{\overline{\triangleright}} y$ and obtain a reduced T_1-groupoid. The authors do not know whether there exist T_1-groupoids such that we do not obtain a reduced T_1-groupoid after removing $x \mathbin{\overline{\triangleleft}} y$ and $x \mathbin{\overline{\triangleright}} y$.

9.3. 2-*Elementary solutions.* We study 2-elementary solutions of the tetrahedron equation with the use of an algebra $(X, *, \circ, \triangleleft, \triangleright)$ with four binary operations, where

$$x \triangleright (y * z) = (x \triangleright y) * (x \triangleright z),$$
$$(x \circ y) \triangleleft z = (x \triangleleft z) \circ (y \triangleleft z),$$
$$(x * y) \circ (z * w) = (x \circ z) * (y \circ w),$$
$$(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z),$$
$$(x * y) \triangleleft z = x \triangleright (y \circ z).$$

In the sequel, we call this algebra the *second tetrahedral* 4*-groupoid* or a T_2*-groupoid*.

Proposition 9.10. *Every* T_2*-groupoid* X *determines a* 2*-elementary solution* (X, R) *of the tetrahedron equation, where*

$$R(x,y,z) = (x,\ x \triangleright (y \circ z),\ z), \quad x,y,z \in X.$$

Proof. Notice that R is a solution if it satisfies the equality in Corollary 9.3. This equality assumes the form

$$x \triangleright ((y \triangleright (t \circ q)) \circ (z \triangleright (p \circ q))) = (x \triangleright (y \circ z)) \triangleright ((x \triangleright (t \circ p)) \circ q).$$

We apply the relations of the T_2-groupoid on the left-hand side of the equality. We obtain

$$\begin{aligned} x \triangleright ((y \triangleright (t \circ q)) \circ (z \triangleright (p \circ q))) &= x \triangleright (((y * t) \triangleleft q) \circ ((z * p) \triangleleft q)) \\ &= x \triangleright (((y * t) \circ (z * p)) \triangleleft q) = (x \triangleright ((y \circ z) * (t \circ p))) \triangleleft q \\ &= ((x \triangleright (y \circ z)) * (x \triangleright (t \circ p))) \triangleleft q = (x \triangleright (y \circ z)) \triangleright ((x \triangleright (t \circ p)) \circ q), \end{aligned}$$

i.e., the required equality holds. □

Assume that there is a 2-elementary solution (X, R) of the tetrahedron equation, where

$$R(x,y,z) = (x, [x,y,z], z), \quad x,y,z \in X.$$

We ask if this solution is determined by a T_2-groupoid.

The following assertion shows that the answer is positive if certain conditions on the ternar $(X, [\cdot,\cdot,\cdot])$ hold.

Proposition 9.11. *Assume that a ternar* $(X, [\cdot,\cdot,\cdot])$ *defines a solution of the tetrahedron equation. Let there exist an element* $c \in X$ *with* $[c,c,c] = c$ *and a function* $\{\cdot\} : X \to X$ *with*

$$\{[c,x,c]\} = [c,\{x\},c] = x, \quad [\{x\},\{y\},c] = \{[x,y,c]\}, \quad [c,\{x\},\{y\}] = \{[c,x,y]\}.$$

We put

$$x * y = [x,y,c], \quad x \circ y = [c,x,y], \quad x \triangleright y = [x,\{y\},c], \quad x \triangleleft y = [c,\{x\},y]$$

for all $x, y \in X$*. Then* $(X, *, \circ, \triangleright, \triangleleft)$ *is a* T_2*-groupoid.*

Proof. It is easy to see that

$$\begin{aligned}(x*y)\circ(z*w)&=[c,[x,y,c],[z,w,c]]=\\&=[[c,x,z],[c,y,w],c]=(x\circ z)*(y\circ w),\\x\triangleright(y*z)&=[x,\{[y,z,c]\},c]=[x,[\{y\},\{z\},c],[c,c,c]]=\\&=[[x,\{y\},c],[x,\{z\},c],c]=(x\triangleright y)*(x\triangleright z),\\(x\circ y)\triangleleft z&=[c,\{[c,x,y]\},z]=[[c,c,c],[c,\{x\},\{y\}],z]=\\&=[c,[c,\{x\},z],[c,\{y\},z]]=(x\triangleleft z)\circ(y\triangleleft z),\\(x\triangleright y)\triangleleft z&=[c,\{[x,\{y\},c]\},z]=\{[c,[x,\{y\},c],[c,z,c]]\}=\\&=\{[[c,x,c],[c,\{y\},z],c]\}=[x,\{[c,\{y\},z]\},c]=x\triangleright(y\triangleleft z),\\x\triangleright(y\circ z)&=[x,\{[c,y,z]\},c]=\{[[c,x,c],[c,y,z],c]\}=\\&=\{[c,[x,y,c],[c,z,c]]\}=[c,\{[x,y,c]\},z]=(x*y)\triangleleft z.\end{aligned}$$

We conclude that all relations of a T_2-groupoid hold. □

We use the operations from the above assertion and construct a mapping

$$(x,y,z)\mapsto(x,\triangleright(y\circ z),z),\quad x,y,z\in X,$$

that gives rise to a solution of the tetrahedron equation.

Question 9.12. Does this solution coincide with the initial solution? In other words, is this true that $[x,y,z]=x\triangleright(y\circ z)$ for all $x,y,z\in X$?

Example 9.13. We consider an algebra $(V,*,\circ,\triangleleft,\triangleright)$ on a vector space V, where

$$\begin{aligned}x*y&:=(1-\beta)x+\beta y,\\x\circ y&:=\beta x+(1-\beta)y,\\x\triangleleft y&:=(1-\beta)x+y,\\x\triangleright y&:=x+(1-\beta)y,\end{aligned}$$

and β is an endomorphism of V. It is easy to see that we obtain a T_2-groupoid. We obtain a 2-elementary solution (V,R) as follows:

$$R(x,y,z)=(x,(1-\beta)x+\beta y+(1-\beta)z,z),\quad x,y,z\in V.$$

On the other hand, if β is an automorphism then we put $c:=0$ and $\{x\}:=\beta^{-1}x$ and construct a T_2-groupoid $(V,*,\circ,\triangleleft,\triangleright)$ from (V,R).

Remark 9.14. It is not difficult to see that the restrictions $(V,*)$ and $(V,\circ)$ of the T_2-groupoid $(V,*,\circ,\triangleleft,\triangleright)$ from the above example form generalized Alexander quandles.

10. VERBAL SOLUTIONS

Let G be a group. A *verbal solution* of the n-simplex equation is a solution (G,R) of the form

$$R(g_1,g_2,\dots,g_n)=(w_1(g_1,g_2,\dots,g_n),w_2(g_1,g_2,\dots,g_n),\dots,w_n(g_1,g_2,\dots,g_n)),$$

where each $w_i=w_i(x_1,x_2,\dots,x_n)$ with $i=1,2,\dots,n$ is a reduced word in a free group $F_n=\langle x_1,\dots,x_n\rangle$. A verbal solution R is said to be *k-elementary* if each word w_i (except for w_k) is equal to x_i.

10.1. *Universal verbal solutions.* If (G,R) is a verbal solution of the n-simplex equation and $\varphi:G\to H$ is a homomorphism such that $(\mathrm{Ker}(\varphi),R|_{\mathrm{Ker}(\varphi)})$ is a solution of the n-simplex equation then we obtain a new solution $(\varphi(G),R^{\varphi})$, where

$$R^{\varphi}(\varphi(g_1),\varphi(g_2),\dots,\varphi(g_n))=(\varphi(g_1'),\varphi(g_2'),\dots,\varphi(g_n'))$$

and

$$R(g_1,g_2,\dots,g_n)=(g_1',g_2',\dots,g_n').$$

As is known, for every group G, there exists an elementary solution $R : G^2 \to G^2$ of the Yang–Baxter equation. We can construct such a solution by taking a quandle on G (for example, either the conjugation quandle $\mathrm{Conj}(G)$ or the core quandle $\mathrm{Core}(G)$). We formulate a general question.

Question 10.1. Let G be a group. Is there a mapping $R : G^n \to G^n$, $n > 2$, that gives rise to nondegenerate (elementary) solutions of the n-simplex equation?

Here by a trivial solution we mean a permutation of components.

In the present section, we consider elementary verbal solutions of the tetrahedron equation on an arbitrary group, which is equivalent to studying solutions on the free group F_6. We recall definitions from the combinatorial group theory. We regard F_n as the free product on n copies of an infinite cyclic group. We represent elements in F_n as reduced words

$$w = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}, \;\; \alpha_j \in \mathbb{Z} \setminus \{0\}, \;\; i_j \neq i_{j+1} \text{ for } j = 1, 2, \dots, k-1. \tag{10.1}$$

Each subword $x_{i_j}^{\alpha_j}$ is called a *syllable*. The number k is called the *syllable length* of w and denoted by $l(w)$. By the i-*syllable length* of w we mean the number of syllables that belong to $\langle x_i \rangle$ (we denote it by $l_i(w)$). For example, if $w = x_3^{-5} x_1^2 x_2^{-7} x_1 x_3^8 \in F_3$ then we have $l(w) = 5$, $l_1(w) = l_3(w) = 2$, and $l_2(w) = 1$.

We also consider cyclic reduced forms of words. Word (10.1) is said to be *cyclic reduced* if either $i_1 \neq i_k$ or both conditions $i_1 = i_k$ and $\alpha_1 \alpha_k > 0$ hold. If a word w is not cyclic reduced then it is of the form $w \equiv u^{-1} w_0 u$, where w_0 is a cyclic reduced subword of w and $\equiv$ denotes the equality of words. Moreover, we have $w^m \equiv u^{-1} w_0^m u$ for every integer m. For example, we have $x_3^{-5} x_1^2 x_2^{-7} x_1 x_3^8 = x_3^{-5} (x_1^2 x_2^{-7} x_1 x_3^3) x_3^5$ and the subword $x_1^2 x_2^{-7} x_1 x_3^3$ is cyclic reduced.

Lemma 10.2. *Let $w = w(x_1, \dots, x_n)$ be a reduced word in a free group F_n. Let $l_j(w) = k > 0$ for a suitable j with $1 \leq j \leq n$. Then $l_j(w^m) \geq k$ for every integer $m \neq 0$. Moreover, the first and last symbols of w coincide with, respectively, the first and last symbols of w^m for every positive m.*

Proof. We assume that $m > 0$. If $m < 0$ then the proof is similar.

We assume that w is cyclic reduced. If $l(w) = l_j(w) = 1$ then $w = x_j^{\alpha}$, where α is a nonzero integer. It is obvious that the required assertion holds. Assume that $l(w) > 1$ and w is a word of the form described in (10.1). If either i_1 or i_k is distinct from j then $l_j(w^m) = km$. If $i_1 = i_k = j$ then $k > 1$ and we have

$$l_j(w^m) = km - (m-1) = m(k-1) + 1.$$

Since $m \geq 2$, we obtain $m(k-1) + 1 \geq 2(k-1) + 1 = 2k - 1 \geq k$.

If w is not cyclic reduced then $w \equiv u^{-1} w_0 u$. If $k = l_j(w) = 2l_j(u) + l_j(w_0)$ then $l_j(w^m) = 2l_j(u) + l_j(w_0^m)$. By the above, we find that $l_j(w_0^m) \geq l_j(w_0)$. Thus, we have $l_j(w^m) \geq 2l_j(u) + l_j(w_0) \geq k$. Let $k = l_j(w) = 2l_j(u) + l_j(w_0) - 1$. This is possible if either both the ultimate syllable of u^{-1} and the initial syllable of w_0 belong to $\langle x_j \rangle$ or both the ultimate syllable of w_0 and the initial syllable of u belong to $\langle x_j \rangle$ (notice that these two conditions cannot hold simultaneously). Then

$$l_j(w^m) = 2l_j(u) + l_j(w_0^m) - 1 \geq 2l_j(u) + l_j(w_0) - 1 \geq k.$$

The second part of the lemma is trivial. □

Lemma 10.3. *Let $w = w(x_1, x_2)$ and $g = g(x_1, x_2, x_3)$ be reduced words in the free group F_5 such that $l(w) = k$, $l_3(g) = m > 0$, and $l(g) > 1$. We denote $n = l_3(w(g(x_1, x_2, x_3), g(x_4, x_5, x_3)))$. Then*

$$n \geq m \quad \text{if } k = 1, \tag{10.2}$$

$$n \geq 2(m-1) + (m-2)(k-2) \quad \text{if } k \geq 2,\ m \geq 2. \tag{10.3}$$

Proof. We assume that

$$w = x_1^{\alpha_1} x_2^{\beta_1} x_1^{\alpha_2} x_2^{\beta_2} \dots x_1^{\alpha_s} x_2^{\beta_s}, \tag{10.4}$$

where $\alpha_j, \beta_j \in \mathbb{Z}$ and only α_1 and β_s may vanish. If $k = 1$ then w is either a power of x_1 or a power of x_2. In this case, we obtain the required assertion from Lemma 10.2.

We prove (10.3). If $m \geq 2$ then there may be only one cancellation of x_3 in $g(x_1, x_2, x_3)^a\, g(x_4, x_5, x_3)^b$. We conclude that

$$l_3(g(x_1, x_2, x_3)^a\, g(x_4, x_5, x_3)^b) \geq (m-1) + (m-1).$$

It is possible to cancel x_3 in $g(x_1, x_2, x_3)^a\, g(x_4, x_5, x_3)^b\, g(x_1, x_2, x_3)^c$ only twice; hence, we have

$$l_3(g(x_1, x_2, x_3)^a\, g(x_4, x_5, x_3)^b\, g(x_1, x_2, x_3)^c) \geq (m-1) + (m-2) + (m-1).$$

We repeat this argument k times and obtain (10.3). ☐

We use the notation from the previous lemma. We obtain the following assertion.

Lemma 10.4. *Let $w = w(x_1, x_2)$, let $l(w) = k \geq 2$, and let*

$$g(x_1, x_2, x_3) \equiv g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2), \quad g_i \not\equiv 1, \quad \alpha \neq 0.$$

If $n = l_3(w(g(x_1, x_2, x_3), g(x_4, x_5, x_3)))$ then $n = k$.

Proof. Assume that

$$w = x_1^{\alpha_1} x_2^{\beta_1} x_1^{\alpha_2} x_2^{\beta_2} \ldots x_1^{\alpha_s} x_2^{\beta_s}, \tag{10.5}$$

where $\alpha_j, \beta_j \in \mathbb{Z}$ and only α_1 and β_s may vanish. We represent

$$\begin{aligned} w(g(x_1, x_2, x_3), g(x_4, x_5, x_3)) &= w(g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2), g_1(x_4, x_5)\, x_3^{\alpha}\, g_2(x_4, x_5)) \\ &= (g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2))^{\alpha_1}\, (g_1(x_4, x_5)\, x_3^{\alpha}\, g_2(x_4, x_5))^{\beta_1}\, (g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2))^{\alpha_2} \\ &\quad \cdot (g_1(x_4, x_5)\, x_3^{\alpha}\, g_2(x_4, x_5))^{\beta_2} \ldots (g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2))^{\alpha_s}\, (g_1(x_4, x_5)\, x_3^{\alpha}\, g_2(x_4, x_5))^{\beta_s}. \end{aligned}$$

Since g_1 and g_2 are nontrivial, cancellations are possible inside either

$$(g_1(x_1, x_2)\, x_3^{\alpha}\, g_2(x_1, x_2))^{\alpha_i} \text{ or } (g_1(x_4, x_5)\, x_3^{\alpha}\, g_2(x_4, x_5))^{\beta_i}$$

and are impossible between these words. We conclude that $n = k$. ☐

We describe verbal 3-elementary solutions of the tetrahedron equation.

Theorem 10.5. *Let $R\colon G^3 \to G^3$ be a verbal 3-elementary solution of the tetrahedron equation for every group G. Then one of the following conditions holds.*

1. *We have $R(x, y, z) = (x, y, yx^{-1})$,*
2. *we have $R(x, y, z) = (x, y, x^{-1}y)$,*
3. *we have $R(x, y, z) = (x, y, w(y, z))$, where $R'(y, z) = (y, w(y, z))$ is a solution of the Yang–Baxter equation for every group G.*

Proof. If

$$R(x, y, z) = (x, y, w(x, y, z)), \quad x, y, z \in G,$$

is a solution of the tetrahedron equation then, by Corollary 9.2, we have

$$w(w(x, y, z), w(x, t, p), w(y, t, q)) = w(y, t, w(z, p, q)). \tag{10.6}$$

We assume that x does not occur in w, i.e., we have $w(x, y, z) = w(y, z)$. In this case, equation (10.6) assumes the form

$$w(w(t, p), w(t, q)) = w(t, w(p, q)).$$

We conclude that $R'(y, z) = (y, w(y, z))$ is a solution of the Yang–Baxter equation for every group G.

We assume that $w(x, y, z)$ depends on x. We consider three possible cases.

Case 1: $w(x, y, z) = w(x, z)$.

In this case, equality (10.6) assumes the form

$$w(w(x, z), w(y, q)) = w(y, w(z, q)). \tag{10.7}$$

It is easy to see that the expression on the left-hand side of (10.7) depends on x, which is a contradiction. We conclude that this case is impossible.

Case 2: $w(x,y,z)$ depends on all generators.

In this case, we may assume that

$$w(x,y,z) \equiv z^{\alpha_1} g_1(x,y) z^{\alpha_2} \dots g_n(x,y) z^{\alpha_{n+1}}, \quad \alpha_j \in \mathbb{Z}, \text{ and only } \alpha_1 \text{ and } \alpha_{n+1} \text{ may vanish.}$$

We rewrite the expression on the left-hand side of (10.6) as follows:

$$(w(y,p,q))^{\alpha_1} g_1(w(x,y,z), w(x,t,p))(w(y,p,q))^{\alpha_2} \dots g_n(w(x,y,z), w(x,t,p))(w(y,p,q))^{\alpha_{n+1}}.$$

Notice that, for every $g_i(x_1,x_2)$, the word $g_i(w(x,y,z), w(x,t,p))$ depends on t and z. Moreover, since $(w(y,p,q))^{\alpha_i}$ depends on q for every nonzero α_i, the word $g_i(w(x,y,z), w(x,t,p))$ is independent of x. By Lemma 10.4, we have either $w = x^\gamma h(y,z)$ or $w = h(y,z)x^\gamma$ for suitable $0 \neq \gamma \in \mathbb{Z}$ and $h(x,y) \in F_2(x,y)$.

Assume that $w = x^\gamma h(y,z)$. If we have $g_{i_0} = y^k$ for some $k \in \mathbb{Z}$ then, by Lemma 10.2, the word $g_{i_0}(w(x,y,z), w(x,t,p))$ depends on x, which is a contradiction. Therefore, we have $h(y,z) = y^\alpha z^\beta$, where $\alpha, \beta \in \mathbb{Z}\backslash\{0\}$. We obtain

$$(x^\gamma y^\alpha z^\beta)^\gamma (x^\gamma t^\alpha p^\beta)^\alpha (y^\gamma p^\alpha q^\beta)^\beta = y^\gamma t^\alpha (z^\gamma p^\alpha q^\beta)^\beta.$$

Since x can be canceled only if $\gamma = -1$ and $\alpha = 1$, we have

$$z^{-\beta} y^{-1} t p^\beta (y^{-1} p q^\beta)^\beta = y^{-1} t (z^{-1} p q^\beta)^\beta.$$

Since $\beta \neq 0$, we arrive at a contradiction.

Similar arguments show that the case in which $w = h(y,z)x^\gamma$ is impossible.

Case 3: $w(x,y,z) = w(x,y)$.

In this case, equality (10.6) assumes the form

$$w(w(x,y), w(x,t)) = w(y,t). \tag{10.8}$$

We know that $l(w(x,y)) = k \geq 2$. We assume that $l_x(w(x,y)) = m \geq 2$. We have

$$l_x(w(w(x,y), w(x,t))) \geq 2(m-1) + (m-2)(k-2) \geq 2(m-1) > 0.$$

We conclude that the word $w(w(x,y), w(x,t))$ depends on x. Thus, we obtain $l_x(w(x,y)) = m = 1$ or, which is equivalent, $w(x,y) \equiv y^a x^b y^c$, where $b \neq 0$.

It is not difficult to prove that $w(x,y) \equiv y^a x^b y^c$ satisfies (10.8) if and only if either $w \equiv x^{-1}y$ or $w \equiv yx^{-1}$. □

Remark 10.6. We consider the permutation $P_{13}(x,y,z) = (z,y,x)$ of the first and third variables and a 3-elementary solution $R(x,y,z) = (x,y,w(x,y,z))$ of the tetrahedron equation. Then

$$P_{13} R P_{13}(x,y,z) = (w(z,y,x), y, z)$$

is a 1-elementary solution of the tetrahedron equation.

We describe verbal 2-elementary solutions of the tetrahedron equation.

Theorem 10.7. *Let $R\colon G^3 \to G^3$ be a verbal 2-elementary solution of the tetrahedron equation for every group G. Then one of the following conditions holds.*

1. *We have $R(x,y,z) = (x, w(x,z), z)$, where either $w \equiv xz$ or $w \equiv zx$,*
2. *we have $R(x,y,z) = (x, w(x,y), z)$, where $R'(x,y) = (x, w(x,y))$ is a solution of the Yang–Baxter equation for every group G,*
3. *we have $R(x,y,z) = (x, w(y,z), z)$, where $R'(y,z) = (y, w(z,y))$ is a solution of the Yang–Baxter equation for every group G.*

Proof. If

$$R(x, y, z) = (x, w(x, y, z), z), \quad x, y, z \in G,$$

is a solution of the tetrahedron equation then, by Corollary 9.3, we have

$$w(w(x, y, z), w(x, t, p), q) = w(x, w(y, t, q), w(z, p, q)). \tag{10.9}$$

We assume that x does not occur in w, i.e., we have $w(x, y, z) = w(y, z)$. In this case, equation (10.9) assumes the form

$$w(w(t, p), q) = w(w(t, q), w(p, q)).$$

We conclude that $R'(y, z) = (y, w(y, z))$ is a solution of the Yang–Baxter equation for every group G.

We assume that $w(x, y, z)$ depends on x. We consider three possible cases.

Case 1: $w(x, y, z) = w(x, z)$.

In this case, equality (10.9) assumes the form

$$w(w(x, z), q) = w(x, w(z, q)). \tag{10.10}$$

Let $l_2(w(x_1, x_2)) = k \geq 1$ and let $l_1(w(x_1, x_2)) = m \geq 1$. By equality (10.10) and Lemma 10.2, we have

$$m = l_x(w(x, w(z, q))) = l_x(w(w(x, z), q)) \geq (k - 1)m.$$

We obtain $l_2(w(x_1, x_2)) \in \{1, 2\}$. Similar arguments prove that $l_1(w(x_1, x_2)) \in \{1, 2\}$. Straightforward calculations yield two solutions; namely, $w(x, z) \equiv xz$ and $w(x, z) \equiv zx$.

Case 2: $w(x, y, z)$ depends on all generators.

Let $m = l_1(w(x_1, x_2, x_3))$ and let $n = l_3(w(x_1, x_2, x_3))$. Then

$$w \equiv \omega_1(x_1, x_2)\, x_3^{\alpha_1}\, \omega_2(x_1, x_2) \ldots x_3^{\alpha_n}\, \omega_{n+1}(x_1, x_2).$$

We assume that $m \geq 3$. It is easy to see that

$$m = l_x(w(x, w(y, t, q), w(z, p, q))) = \sum_{i=1}^{n+1} l_x(\omega_i(w(x, y, z), w(x, t, r))).$$

We assume that there exists a subword $\omega_j(x_1, x_2)$ of w such that $k = l(\omega_j(x_1, x_2)) > 1$. By Lemma 10.3, we have

$$m \geq l_x(\omega_j(w(x, y, z), w(x, t, r))) \geq 2(m - 1) + (m - 2)(k - 2) \geq 2m - 2 > m.$$

Therefore the syllable length of each subword $\omega_i(x_1, x_2)$ is equal to one, i.e., we have either $\omega_i(x_1, x_2) \equiv x_1^{\alpha_i}$ or $\omega_i(x_1, x_2) \equiv x_2^{\beta_i}$. By Lemma 10.2, we have $l_x(\omega_j(w(x, y, z), w(x, t, r))) \geq m$. We obtain $l + 1 = 1$ in the above sum; however, the word w depends on x_3. We conclude that $m \in \{1, 2\}$.

Similar arguments prove that $n \in \{1, 2\}$. A finite set of possibilities for w remains. Straightforward verification shows that there are no solutions among them.

Case 3: $w(x, y, z) = w(x, y)$.

In this case, equality (10.6) assumes the form

$$w(w(x, y), w(x, t)) = w(x, w(y, t)). \tag{10.11}$$

We conclude that $R'(x, y) = (x, w(x, y))$ is a solution of the Yang–Baxter equation for every group G.□

The above results show that there are no nontrivial invertible elementary verbal solutions of the tetrahedron equation on free non-Abelian groups. The authors do not know whether such nonelementary solutions exist. On the other hand, Korepanov constructed a nontrivial invertible solution of the 4-simplex equation on a free non-Abelian group.

Example 10.8 (I. G. Korepanov)**.** Let G be a group. We fix elements $a, b \in G$. Then the mapping

$$(x, y, z, w) \mapsto (yw^{-1}a, xbz, w, z), \quad x, y, z, w \in G,$$

determines a solution of the 4-simplex equation on G. It is easy to see that this solution is verbal if $a = b = 1$.

Definition 10.9. We consider the endomorphism $R : F_n \to F_n$ of the free group defined by the rule

$$R(x_1, x_2, \ldots, x_n) = (w_1(x_1, x_2, \ldots, x_n), w_2(x_1, x_2, \ldots, x_n), \ldots, w_n(x_1, x_2, \ldots, x_n))$$

on the generators. Let $\mathcal{V}$ be a group variety. We say that R determines a *universal verbal $\mathcal{V}$-solution* of the n-simplex equation if (G, R) is a solution for every group $G \in \mathcal{V}$. If $\mathcal{V}$ is the variety of Abelian groups then we speak about universal verbal Abelian solutions, if $\mathcal{V}$ is a variety of nilpotent groups then we speak about universal verbal nilpotent solutions, etc.

10.2. *Verbal solutions in Abelian groups.* We consider the free Abelian group of rank n and find universal verbal Abelian solutions of the n-simplex equation. There is a one-to-one correspondence between verbal solutions of the n-simplex equation on the free Abelian group $\mathbb{Z}^n$ and linear solutions over the ring $\mathbb{Z}$. Each linear solution R over $\mathbb{Z}$ can be represented by a matrix, i.e., we have

$$R(x) = Mx, \quad x \in \mathbb{Z}^n,\ M \in \mathcal{M}_n(\mathbb{Z}).$$

Proposition 10.10. *If a mapping R is determined by a matrix M and is a solution of the n-simplex equation then the mapping determined by the transpose matrix M^T is a solution of the n-simplex equation too.*

In several cases, solutions of the n-simplex equation are known. For example, all linear bijective solutions of the 2-simplex, 3-simplex, and 4-simplex equation are listed in [21]. We list all universal Abelian solutions in these cases.

Proposition 10.11. *Let G be an Abelian group and let $\alpha, \beta \in \mathbb{Z}$. Then each universal verbal solution of the Yang–Baxter equation is determined by one of the following mappings, where $a, b \in G$:*

$$\begin{aligned}
&(a, b) \mapsto (a^\alpha, b^\beta),\\
&(a, b) \mapsto (b, a),\\
&(a, b) \mapsto (a^\alpha b^{1-\alpha\beta}, b^\beta),\\
&(a, b) \mapsto (a^\alpha, a^{1-\alpha\beta} b^\beta).
\end{aligned}$$

Proposition 10.12. *Let G be an Abelian group and let $\alpha, \beta, \gamma \in \mathbb{Z}$. Then each universal verbal solution of the tetrahedron equation is either the k-amalgam of solutions of the Yang–Baxter and* 1*-simplex equations or is obtained from mappings in the list*

$$\begin{aligned}
&(a, b, c) \mapsto (a^\alpha, a, a^{-\beta} b c^\beta),\\
&(a, b, c) \mapsto (b, a, a^{-\alpha} b c^\alpha),\\
&(a, b, c) \mapsto (a^\alpha b^{1-\alpha\beta} c^{\alpha(\beta\gamma-1)}, b^\beta c^{1-\beta\gamma}, c^\gamma),\\
&(a, b, c) \mapsto (a^\alpha b^{1-\alpha\beta} c^{\gamma(\alpha\beta-1)}, b^\beta c^{1-\beta\gamma}, c^\gamma),
\end{aligned}$$

where $a, b, c \in G$, by conjugation (*see Proposition* 4.4) *and transposition* (*see Proposition* 10.10).

Proposition 10.13. *Let G be an Abelian group and let $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Then each universal verbal solution of the 4-simplex equation is either the k-amalgam of solutions of the m-simplex equation with $m < 4$ or is obtained from mappings in the list*

$$
\begin{aligned}
&(a,b,c,d) \mapsto (bd^{-1}, ac, d, c),\\
&(a,b,c,d) \mapsto (bc^{-\alpha}d^{\alpha\beta-1}, ac, c^{\alpha}d^{1-\alpha\beta}, d^{\beta}),\\
&(a,b,c,d) \mapsto (b, a, a^{-\alpha}bc^{\alpha}, a^{\alpha\beta-1}c^{1-\alpha\beta}d^{\beta}),\\
&(a,b,c,d) \mapsto (bc^{-\alpha}d^{\alpha\beta}, acd^{-\beta}, c^{\alpha}d^{1-\alpha\beta}, d^{\beta}),\\
&(a,b,c,d) \mapsto (b, a, a^{-\alpha}bc^{\alpha}, a^{\alpha\beta}b^{-\beta}c^{1-\alpha\beta}d^{\beta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}bc^{-\alpha}, c, b, b^{-\beta}cd^{\beta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}, bcd^{-\beta}, a^{-\alpha}bd, d^{\beta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}bc^{-\alpha}d^{\alpha\beta}, cd^{-\beta}, bd, d^{\beta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}b^{1-\alpha\beta}c^{\gamma(\alpha\beta-1)}d^{\gamma\delta(1-\alpha\beta)}, b^{\beta}c^{1-\beta\gamma}d^{\delta(\beta\gamma-1)}, c^{\gamma}d^{1-\gamma\delta}, d^{\delta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}b^{1-\alpha\beta}c^{\alpha(\beta\gamma-1)}d^{\alpha\delta(1-\beta\gamma)}, b^{\beta}c^{1-\beta\gamma}d^{\delta(\beta\gamma-1)}, c^{\gamma}d^{1-\gamma\delta}, d^{\delta}),\\
&(a,b,c,d) \mapsto (a^{\alpha}b^{1-\alpha\beta}c^{\alpha(\beta\gamma-1)}d^{\alpha\beta(1-\gamma\delta)}, b^{\beta}c^{1-\beta\gamma}d^{\beta(\gamma\delta-1)}, c^{\gamma}d^{1-\gamma\delta}, d^{\delta}),
\end{aligned}
$$

where $a, b, c, d \in G$, by conjugation (see Proposition 4.4) and transposition (see Proposition 10.10).

11. QUESTIONS FOR FURTHER STUDY

In conclusion, we formulate open questions for further study.

1. Find verbal universal solutions of the n-simplex equation in the class of all groups.
2. A connection is known between the braid group B_3 and the Yang–Baxter equation. Which groups correspond to the n-simplex equation?
3. Multi-switches were introduced in [3, 4]. Is it possible to use them for constructing solutions of the parametric Yang–Baxter equation?

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